

Duality in percolation via outermost boundaries I: Bond Percolation

Ghurumuruhan Ganesan *

New York University, Abu Dhabi

Abstract

Tile \mathbb{R}^2 into disjoint unit squares $\{S_k\}_{k \geq 0}$ with the origin being the centre of S_0 and say that S_i and S_j are star adjacent if they share a corner and plus adjacent if they share an edge. Every square is either vacant or occupied. Outermost boundaries of finite star and plus connected components frequently arise in the context of contour analysis in percolation and random graphs. In this paper, we derive the outermost boundaries for finite star and plus connected components using a piecewise cycle merging algorithm. For plus connected components, the outermost boundary is a single cycle and for star connected components, we obtain that the outermost boundary is a connected union of cycles with mutually disjoint interiors. As an application, we use the outermost boundaries to give an alternate proof of mutual exclusivity of left right and top bottom crossings in oriented and unoriented bond percolation.

Key words: Star and plus connected components, outermost boundary, union of cycles, left right and top bottom crossings.

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*E-Mail: gganesan82@gmail.com

1 Introduction

Tile \mathbb{R}^2 into disjoint unit squares $\{S_k\}_{k \geq 0}$ with origin being the centre of S_0 . Every square in $\{S_k\}$ is assigned one of the two states, occupied or vacant and the square S_0 containing the origin is always occupied. For $i \neq j$, we say that S_i and S_j are *adjacent* or *star adjacent* if they share a corner between them. We say that S_i and S_j are *plus adjacent*, if they share an edge between them. Here we follow the notation of Penrose (2003).

The structure of the outermost boundary of finite components is crucial for contour analysis problems of percolation (Grimmett (1999), Ganesan (2014)) and random graphs (Penrose (2003), Ganesan (2013)). For plus connected components, the boundaries have been well studied before (see for example, Penrose (2003)) and it is also relatively easy to visualize that the boundary must be a single cycle. For star connected components, visualization is a bit difficult since there are many possible candidates for the outermost boundary. Below, we give a definition of outermost boundary that holds for both star and plus connected components and derive the structure of the outermost boundary for the star connected component. The corresponding result for the plus connected component is then obtained as a corollary.

Before we present our results, we briefly enumerate some recent literature containing the related duality problem of percolation. Timár (2013) uses separating sets in equivalence class of infinite paths to study duality in slightly more general locally finite graphs and Penrose (2003) uses unicoherence and topological arguments to investigate plus connected components. Bollobás and Riordan (2006) use a step by step construction for obtaining the outermost boundary in a slightly related problem in bond percolation. In essence, most of the proofs use either substantial topology or infinite graphs.

Our aim in this paper is two fold. We derive the outermost boundary structure for *both* star and plus connected components. We then use the finite graph theoretic structure of the outermost boundaries to provide an alternate proof of mutual exclusivity between left right and top bottom crossings in both unoriented and oriented crossings in percolation.

Model Description

We first discuss star connected components. We say that the square S_i is connected to the square S_j by a *star connected S -path* if there is a sequence of distinct squares $(Y_1, Y_2, \dots, Y_t), Y_l \subset \{S_k\}, 1 \leq l \leq t$ such that Y_l is star

adjacent to Y_{l+1} for all $1 \leq l \leq t-1$ and $Y_1 = S_i$ and $Y_t = S_j$. If all the squares in $\{Y_l\}_{1 \leq l \leq t}$ are occupied, we say that S_i is connected to S_j by an *occupied* star connected S -path.

Let $C(0)$ be the collection of all occupied squares in $\{S_k\}$ each of which is connected to the square S_0 by an occupied star connected S -path. We say that $C(0)$ is the star connected occupied component containing the origin. Throughout we assume that $C(0)$ is finite and let $\{J_k\}_{1 \leq k \leq M} \subset \{S_j\}$ be the set of all the occupied squares belonging to the component $C(0)$. In this paper, we study the outermost boundary for finite star connected components containing the origin and by translation, the results hold for arbitrary finite star connected components.

It is possible that a star connected component has multiple choices for the outermost boundary. Consider for example, the component consisting of the union of four squares S_a, S_b, S_c and S_d each sharing exactly one edge with the square S_0 containing the origin. If we consider each square itself as a cycle, then the union of the edges of these four cycles could itself be considered as the boundary. On the other hand, it is also possible to consider the “bigger” cycle consisting of edges in $(S_a \cup S_b \cup S_c \cup S_d) \setminus S_0$ as the boundary.

To avoid ambiguities as described in the above paragraph, we give below a formal definition of outermost boundary. We first have a few preliminary definitions. Let G_0 be the graph with vertex set being the set of all corners of the squares of $\{S_k\}$ in the component $C(0)$ and edge set consisting of the edges of the squares of $\{S_k\}$ in $C(0)$. Two vertices in the graph G_0 are said to be adjacent if they share an edge between them. Two edges in G_0 are said to be adjacent if they share an endvertex between them.

Let $P = (e_1, e_2, \dots, e_t)$ be a sequence of distinct edges in G_0 . We say that P is a *path* if e_i and e_{i+1} are adjacent for every $1 \leq i \leq t-1$. Let a be the endvertex of e_1 not common to e_2 and let b be the endvertex of e_t not common to e_{t-1} . The vertices a and b are the *endvertices* of the path P .

We say that P is a *self avoiding path* if the following three statements hold: The edge e_1 is adjacent only to e_2 and no other $e_j, j \neq 2$. The edge e_t is adjacent only to e_{t-1} and no other $e_j, j \neq t-1$. For each $1 \leq i \leq t-1$, the edge e_i shares one endvertex with e_{i-1} and another endvertex with e_{i+1} and is not adjacent to any other edge $e_j, j \neq i-1, i+1$.

We say that P is a *circuit* if $(e_1, e_2, \dots, e_{t-1})$ forms a path and the edge e_t shares one endvertex with e_1 and another endvertex with e_{t-1} . We say that P is a *cycle* if $(e_1, e_2, \dots, e_{t-1})$ is a self avoiding path and the edge e_t shares one endvertex with e_1 and another endvertex with e_{t-1} and does not share an

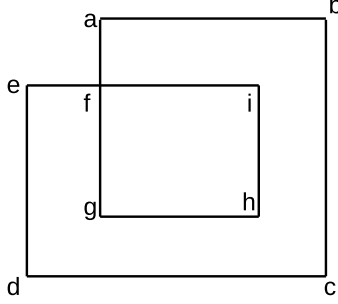


Figure 1: The sequence of vertices $abcdefghi fa$ form a circuit but not a cycle.

endvertex with any other edge e_j , $2 \leq j \leq t - 2$. We emphasize here that we consider only cycles that do not intersect themselves. In other words, every vertex in a cycle C is adjacent to *exactly* two edges of C . For example, the sequence of edges formed by the vertices $abcdefghi fa$ in Figure 1 is a circuit but not a cycle. The sequence of vertices $abcdefa$ forms a cycle.

Any cycle C contains at least four edges and divides the plane \mathbb{R}^2 into two disjoint connected regions. As in Bollobás and Riordan (2006), we denote the bounded region to be the *interior* of C and the unbounded region to be the *exterior* of C . We use cycles to define the outermost boundary of star connected components.

Let e be an edge in the graph G_0 defined above. We say that e is adjacent to a square S_k if it is one of the edges of S_k . We say that the edge e is contained in the *interior* of the cycle C if both the squares in $\{S_k\}$ containing e as an edge, lie in the interior of C . An analogous definition holds for edges in the exterior of C . We say that e is a *boundary edge* if it is adjacent to a vacant square and is also adjacent to an occupied square of the component $C(0)$. Let C be any cycle of edges in G_0 . We have the following definition.

Definition 1. *We say that the edge e in the graph G_0 is an outermost boundary edge of the component $C(0)$ if the following holds true for every cycle C in G_0 : either e is an edge in C or e belongs to the exterior of C .*

We define the outermost boundary ∂_0 of $C(0)$ to be the set of all outermost boundary edges of G_0 .

Thus outermost boundary edges cannot be contained in the interior of

any cycle in the graph G_0 . Our main result is the following.

Theorem 1. *Suppose $C(0)$ is finite. The outermost boundary ∂_0 of $C(0)$ is the union of a unique set of cycles C_1, C_2, \dots, C_n in G_0 with the following properties:*

- (i) *Every edge in $\cup_{1 \leq i \leq n} C_i$ is an outermost boundary edge.*
- (ii) *The graph $\cup_{1 \leq i \leq n} C_i$ is a connected subgraph of G_0 .*
- (iii) *If $i \neq j$, the cycles C_i and C_j have disjoint interiors and have at most one vertex in common.*
- (iv) *Every occupied square $J_k \in C(0)$ is contained in the interior of some cycle C_j .*
- (v) *If $e \in C_j$ for some j , then e is a boundary edge adjacent to an occupied square of $C(0)$ contained in the interior of C_j and also adjacent to a vacant square lying in the exterior of all the cycles in ∂_0 .*

Moreover, there exists a circuit C_{out} containing every edge of $\cup_{1 \leq i \leq n} C_i$.

The outermost boundary ∂_0 is therefore also an Eulerian graph with C_{out} denoting the corresponding Eulerian circuit (for definitions, we refer to Chapter 1, Bollobás (2001)). We remark that the above result also provides a more detailed justification of the statement made about the outermost boundary and the corresponding circuit in the proof of Lemma 3 of Ganesan (2013).

The proof technique of the above result also allows us to obtain the outermost boundary for plus connected components. We recall that squares S_i and S_j are *plus adjacent* if they share an edge between them. We say that the square S_i is connected to the square S_j by a *plus connected S -path* if there is a sequence of distinct squares $(Y_1, Y_2, \dots, Y_t), Y_l \subset \{S_k\}, 1 \leq l \leq t$ such that Y_l is plus adjacent to Y_{l+1} for all $1 \leq l \leq t-1$ and $Y_1 = S_i$ and $Y_t = S_j$. If all the squares in $\{Y_l\}_{1 \leq l \leq t}$ are occupied, we say that S_i is connected to S_j by an *occupied plus connected S -path*.

Let $C^+(0)$ be the collection of all occupied squares in $\{S_k\}$ each of which is connected to the square S_0 by an occupied plus connected S -path. We say that $C^+(0)$ is the plus connected occupied component containing the origin. Throughout we assume that $C^+(0)$ is finite. Let G_0^+ be the graph with vertex set being the set of all corners of the squares of $\{S_k\}$ in $C^+(0)$ and edge set consisting of the edges of the squares of $\{S_k\}$ in $C^+(0)$.

Every plus connected component is also a star connected component and so the definition of outermost boundary edge in Definition 1 holds for the component $C^+(0)$ with G_0 replaced by G_0^+ . We have the following result.

Theorem 2. *Suppose $C^+(0)$ is finite. The outermost boundary ∂_0^+ of $C^+(0)$ is unique cycle in G_0^+ with the following properties:*

- (i) *All squares of $C^+(0)$ are contained in the interior of ∂_0^+ .*
- (ii) *Every edge in ∂_0^+ is a boundary edge adjacent to an occupied square of $C^+(0)$ contained in the interior of ∂_0^+ and a vacant square in the exterior.*

This is in contrast to star connected components which may contain multiple cycles in the outermost boundary.

To prove Theorems 1 and 3, we use the following intuitive result about merging cycles. Let G be the graph with vertex set being the corners of the squares $\{S_k\}_{k \geq 0}$ and edge set being the edges of the squares $\{S_k\}_{k \geq 0}$.

Theorem 3. *Let C and D be cycles in the graph G that have more than one vertex in common. There exists a unique cycle E consisting only of edges of C and D with the following properties:*

- (i) *The interior of E contains the interior of both C and D .*
- (ii) *If an edge e belongs to C or D , then either e belongs to E or is contained in its interior.*

Moreover, if D contains at least one edge in the exterior of C , then the cycle E also contains an edge of D that lies in the exterior of C .

The above result essentially says that if two cycles intersect at more than one point, there is an innermost cycle containing both of them in its interior. We provide an iterative piecewise algorithmic construction for obtaining the cycle E , analogous to Kesten (1980) for crossings, in Section 2.

Bond Percolation

In this section, we use the structure of the outermost boundaries derived in the previous subsection to give alternate proof for mutual exclusivity of left right and top bottom crossings in oriented and unoriented bond percolation. We first discuss for unoriented bond percolation and then consider oriented bond percolation. We also remark that most of the existing approaches for obtaining the mutual exclusivity mainly use some version of interface graphs (Bollobás and Riordan (2006)) that is usually obtained via a step by step procedure. Our method uses the outermost boundaries to directly obtain the presence of the closed top bottom dual crossing in the absence of open left right crossings. For more material on left right and top bottom crossings we refer to Bollobás and Riordan (2006).

We recall that G is the graph with vertex set as the set of corners of the squares in $\{S_k\}$. The edge set is the set of edges of the squares in $\{S_k\}$. Let G_d be the graph obtained by shifting the graph G by $(\frac{1}{2}, \frac{1}{2})$. The graph G_d tiles \mathbb{R}^2 into disjoint unit squares $\{W_k\}$ such that $W_k = S_k + (\frac{1}{2}, \frac{1}{2})$. We say that the graph G is the *dual graph* of G_d .

Consider bond percolation in the graph G_d where every edge is either open or closed. By construction every edge f of the graph G_d intersects perpendicularly a unique dual edge $e = e(f)$ of G . We say that e is open if and only if f is open. Let $R = [0, m] \times [0, n]$ be the $m \times n$ rectangle in the graph G_d containing exactly mn edges of G_d . Let e be an edge with endvertices u_1 and u_2 . We say that edge e lies in the *interior* of R if the following property holds. For $i = 1, 2$, the endvertex u_i either belongs to the boundary of R or lies in the interior of R .

Let $R_{left} = \{0\} \times [0, n]$ and $R_{right} = \{m\} \times [0, n]$ be the left and right edges of R , respectively. We say that a self avoiding path $P = (e_1, \dots, e_k)$ is a left right crossing for R if the following three conditions hold:

- (a) The edge e_1 contains exactly one endvertex in R_{left} and no other edge in P intersects R_{left} .
- (b) The edge e_k contains exactly one endvertex in R_{right} and no other edge in P intersects R_{right} .
- (c) Every edge $e_j, 2 \leq j \leq k - 1$ have both their endvertices in R .

If every edge in P is open we say that P is an open left right crossing. An analogous definition holds for top bottom and open top bottom crossings of R .

We have a similar definition for the dual crossing. We say that the dual edge $f \in G$ intersects the left (right) edge of R if f intersects some edge of G_d contained in the left (right) edge of R . We define a dual left right crossing of the rectangle R as follows. We say that a self avoiding path $P_d = (f_1, \dots, f_m)$ in the graph G is a *dual left right crossing* for R if the edge f_1 intersects R_{left} , the edge f_m intersects R_{right} and every other edge $f_j, 2 \leq j \leq m - 1$ have both their endvertices in the interior of the rectangle R . As before, we say that P_d is an *open* dual left right crossing if every edge in P_d is open. An analogous definition holds for dual top bottom and open dual top bottom crossings of R .

We have the following result regarding left right and top bottom crossings.

Theorem 4. *One of the following two events always occurs but not both:*

- (i) *The rectangle R contains an open left right crossing.*

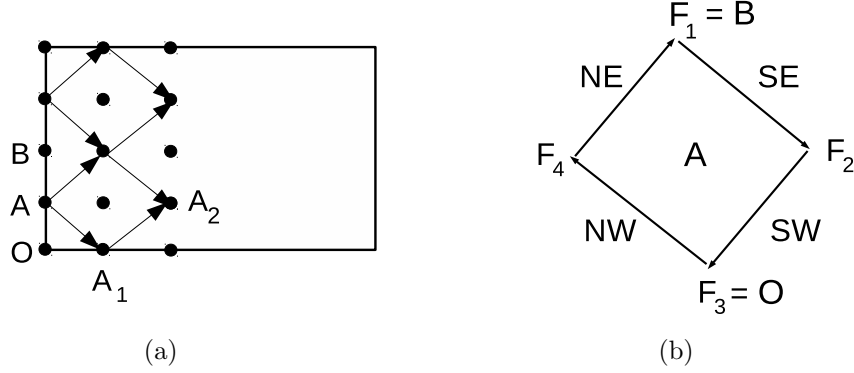


Figure 2: (a) The oriented bond percolation model in the rectangle R . Here O is the origin, $A = (0, 1)$, $B = (0, 2)$, $A_1 = (1, 0)$ and $A_2 = (2, 1)$. (b) The square S_A'' with centre A forming an oriented cycle is shown with the labels for the corresponding orientations.

- (ii) The rectangle R contains a closed dual top bottom crossing.
By rotating rectangles, we also have that one of the following two events always occurs but not both:
- (iii) The rectangle R contains an open left right dual crossing.
- (iv) The rectangle R contains a closed top bottom crossing.

Oriented bond percolation

As before, we consider the $m \times n$ rectangle $R = [0, m] \times [0, n]$. We consider the oriented edge model in R where oriented edges are present as follows.

(b1) If $i = 0$ or i is even, then there is an oriented edge from (i, j) to $(i+1, j+1)$ and from (i, j) to $(i+1, j-1)$, for $1 \leq j \leq n-1$, j odd. If n is odd, there is only the oriented edge from (i, n) to $(i+1, n-1)$.

(b2) If i is odd, then for $2 \leq j \leq n-1$, j even, there is an oriented edge from (i, j) to $(i+1, j+1)$ and from (i, j) to $(i+1, j-1)$. For $j = 0$, there is only the oriented edge from $(i, 0)$ to $(i+1, 1)$ and if n is even, there is only the oriented edge from (i, n) to $(i+1, n-1)$.

We refer to Figure 2(a) for illustration where we have drawn part of the oriented bond percolation model for $i = 0$ and $i = 1$.

Let \mathcal{F}_{or} denote the set of all oriented edges with both endvertices in

the rectangle R as described above. Let $e_i \in \mathcal{F}_{or}, i = 1, 2$ be two oriented edges in the rectangle R and let x_i and y_i denote the non arrow and arrow endvertices of e_i , respectively. We say that (e_1, e_2) is a *consistent pair* if the edges share an endvertex and the arrow end y_1 of e_1 coincides with the non arrow endvertex x_2 of e_2 ; i.e., $y_1 = x_2$.

We say that a sequence of distinct oriented edges $P = (e_1, e_2, \dots, e_k)$ form an *oriented path* if for every $1 \leq i \leq k - 1$, the pair of edges (e_i, e_{i+1}) is a consistent pair. Let x be the endvertex of e_1 not common with e_2 and let y be the endvertex of e_k not common with e_{k-1} . We say that x and y are the *endvertices* of P . If P is any oriented path, one endvertex of P is a non arrow endvertex and the other endvertex is an arrow endvertex. In Figure 2(a), the pair of edges between the points AA_1 and A_1A_2 are consistent and form a oriented path with endvertices A and A_2 .

Let $R_{left} = \{0\} \times [0, n]$ and $R_{right} = \{m\} \times [0, n]$ respectively denote the left and right edges of the rectangle R . An oriented path having one non arrow endvertex in R_{left} and another arrow endvertex in R_{right} is called an *oriented left right crossing of R* . To every oriented edge $e \in \mathcal{F}_{or}$, we assign one of the two following states: open or closed. If every edge in an oriented left right crossing P of R is open, we say that P is an open oriented left right crossing of R .

The next step is to define the *dual lattice* and we follow the notation of Durrett (1984). We tile \mathbb{R}^2 into disjoint 1×1 squares $\{S''_z\}_{z \in \mathbb{Z}^2}$, so that if $z = (i, j)$, then S''_z has endvertices $(i, j-1)$, $(i+1, j)$, $(i, j+1)$ and $(i-1, j)$. We orient the edges of S''_z in such a way that they form a clockwise oriented cycle; i.e. an oriented path with coincident endvertices. Every edge in S''_z is therefore one of the four following types: type 1(\nearrow), type 2(\nwarrow), type 3(\searrow) and type 4(\swarrow). According to the orientations, we also call them as NE , NW , SE and SW arrows, respectively, representing north east, north west, south east and south west directions. Edges belonging to the square $\{S''_z\}_{z \in R}$ are called *dual edges*. In Figure 2(b) we have illustrated the square S''_A with centre $A = (0, 1)$.

Dual edges can lie in the interior or the exterior of the rectangle R . Every dual edge lying in the interior of R is assigned one of the two states open or closed as follows. Let f be a dual edge contained in the interior of R ; i.e., no endvertex of f lies in the exterior of R . The edge f intersects a unique edge $e = e(f)$ belonging to the original percolation model described in Figure 2. We say that f is open if e is open and f is closed if e is closed. Thus open or closed dual edges necessarily lie in the interior of R .

We say that a dual oriented path $\Pi = (e_1, e_2, \dots, e_k)$ is a *dual oriented top bottom crossing* of the rectangle R if the following properties (a1) – (a3) hold.

- (a1) The first edge e_1 has its non-arrow endvertex in the top edge R_{top} of R .
- (a2) The last edge e_k has its arrow endvertex at the bottom edge R_{bottom} of R .
- (a3) If u is an endvertex of an edge $e \in \Pi$, then either u lies on the boundary of R or lies in the interior of R .

If every dual edge in the path Π is closed, we say that Π is a *closed dual oriented top bottom crossing*.

We have the following result.

Theorem 5. *One of the following two events always occurs but not both:*

- (i) *The rectangle R contains an open oriented left right crossing.*
- (ii) *The rectangle R contains a closed dual oriented top bottom crossing.*

The paper is organized as follows: In Section 2, we prove the result Theorem 3 regarding merging of two cycles using a piecewise merging algorithm. In Section 3, we prove Theorems 1 and 2 regarding the outermost boundary of star and plus connected components, respectively. In Section 4, we prove Theorem 4 regarding the mutual exclusivity of left right and top bottom crossings in unoriented bond percolation. Finally, in Sections 5 and 6, we prove the corresponding results for oriented bond percolation. In Section 5, we obtain preliminary properties regarding the outermost boundary with orientation needed for proving Theorem 5. In Section 6, we then prove Theorem 5.

2 Proof of Theorem 3

Proof of Theorem 3: If every edge of the cycle C either belongs to D or is contained in the interior of D , then the desired cycle $E = D$. Similarly, if every edge of the cycle D either belongs to C or is contained in the interior of C , then $E = C$. In what follows, we suppose that the cycle C contains at least one edge in the exterior of D and similarly, the cycle D also contains at least one edge in the exterior of C .

To merge the cycles C and D , we use bridges. Let $P \subset C$ be any path of edges contained in the cycle C . We say that $P = B(P, D)$ is a *bridge* for cycle D if the endvertices of the path P belong to D and every other vertex

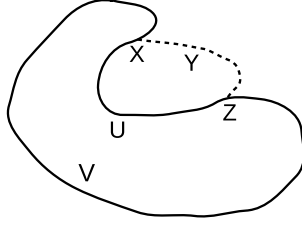


Figure 3: Merging the cycle $XUZVX$ with the segment XYZ .

in P lies in the exterior of D . In particular, every edge of P lies in the exterior of the cycle D .

We start with cycles $F_1 = D = (e_1, \dots, e_s)$ and $C = (f_1, \dots, f_t)$. In the first step, we identify a bridge P_1 for the cycle F_1 contained in the cycle C . In the second step, we merge the cycle F_1 with the bridge P_1 to get a new cycle F_2 . We then repeat the above procedure with the cycle F_2 and continue this process iteratively until all the edges of the cycle C exterior to the cycle D are exhausted. The final cycle obtained is the desired cycle E .

Step 1: Extracting the bridge P_1 from the cycle C

For $1 \leq j \leq t - 1$, let v_j be the endvertex common to the edges f_j and f_{j+1} and let v_0 be the endvertex common to edges f_1 and f_t . Thus the edge f_j in the cycle C has endvertices v_{j-1} and v_j for $1 \leq j \leq t$.

Let j_1 be the least index $j \geq 1$ so that the edge $f_j \in C$ lies in the exterior of the cycle D . We then have that one endvertex v_{j_1-1} of f_{j_1} belongs to D and the other endvertex v_{j_1} lies in the exterior of D . Without loss of generality, we assume that $j_1 = 1$.

If every edge $f_k, k \geq 2$, has both its endvertices in the exterior of the cycle F_1 , then all the edges of the cycle C lie in the exterior the cycle F_1 . In particular, since the cycle D is contained in the interior of the cycle F_1 , we have that every vertex of the cycle C apart from the vertex v_0 lies in the exterior of the cycle D . This is a contradiction since we assume that C and D have more than one vertex in common. Therefore there exists an edge $f_k, k \geq 2$ such that the endvertex $v_k \neq v_0$ of the edge f_k belongs to D . Let f_{k_1} be the edge with the least such index. We then have that the endvertex v_{k_1-1} of f_{k_1} lies in the exterior of D . The path of edges $P_1 = (f_1, f_2, \dots, f_{k_1}) \subset C$

is therefore a bridge for the cycle D with endvertices v_0 and v_{k_1} .

In Figure 3, the cycle F_1 is represented as $XUZVX$. The bridge $P_1 = XYZ$ with X representing the endvertex v_0 and Z representing the endvertex v_{k_1} .

Step 2: Merging the bridge P_1 with the cycle F_1

The bridge P_1 has endvertices v_0 and v_{k_1} . Both v_0 and v_{k_1} also belong to the cycle F_1 . Let $F_1 = Q_1 \cup R_1$ be the union of two paths where both Q_1 and R_1 have endvertices v_0 and v_{k_1} . Let $G_1 = P_1 \cup Q_1$ and $H_1 = P_1 \cup R_1$ be the two cycles obtained by the union of the bridge P_1 with the two subpaths Q_1 and R_1 of the cycle D . Exactly one of the cycles, say H_1 , contains the cycle D in the interior and the cycles G_1 and D have mutually disjoint interiors.

In Figure 3, the paths Q_1 and R_1 are respectively represented by the segments XUZ and XVZ . The union of the paths $P_1 \cup R_1$ is the cycle H_1 which contains the cycle D in its interior. We define H_1 to be the cycle obtained at the end of the first iteration. We have the following properties regarding the cycle H_1 .

- (a1) The cycle H_1 contains only edges from the cycles C and D .
- (a2) Every edge of the cycle D either belongs to H_1 or is contained in the interior of H_1 . Therefore the interior of the cycle D is contained in the interior of the cycle H_1 .
- (a3) The cycle H_1 contains at least one edge of D lying in the exterior of the cycle C .

Proof of (a1) – (a3) for the cycle H_1 : The properties (a1) – (a2) are true by construction and the property (a3) is true since every edge in the bridge $\emptyset \neq P_1 \subset H_1$ lies in the exterior of the cycle D . ■

To proceed to the next step of the iteration, we set $F_2 = H_1$ and repeat the above procedure with F_1 replaced by F_2 . Again the cycle H_2 obtained at the end of the iteration step satisfies (a1) – (a3). This procedure continues for a finite number of steps until we obtain a final cycle H_n .

It remains to see that the cycle H_n is the desired cycle E mentioned in the statement of the Theorem. Since H_n satisfies property (a1), we have that the cycle H_n contains only edges from the cycles C and D . Since H_n also satisfies property (a3), we have the property (ii) in the statement of the theorem is true.

To see that (i) is true, we argue as follows. By definition, the interior of the cycle D is contained in the interior of the cycle H_n . If there exists

an edge of the cycle C lying in the exterior of the cycle H_n , then we could extract another bridge from cycle C and the procedure above would not have terminated. Thus every edge in the cycle C either belongs to H_n or is contained in the interior of the cycle H_n . This proves property (i).

To see the uniqueness of the cycle H_n , suppose that there is another cycle $K \neq H_n$ that satisfies the statement of the Theorem. Without loss of generality, the cycle K contains an edge e in the exterior of H_n . The edge $e \in C \cup D$ and suppose $e \in C$. This means that at least one edge of C lies in the exterior of H_n , a contradiction since H_n satisfies property (i). ■

3 Proof of Theorems 1 and 2

We first prove Theorem 1 and obtain Theorem 2 as a Corollary. The first step in the proof of Theorem 1 is to obtain large cycles surrounding each occupied square in $C(0)$. We recall that G_0 is the graph with vertex set being the set of all corners of the squares $\{S_k\}_{k \geq 0}$ in $C(0)$ and edge set consisting of the edges of the squares $\{S_k\}_{k \geq 0}$ in $C(0)$. Also $\{J_k\}_{1 \leq k \leq M} \subset \{S_j\}$ denotes the set of occupied squares belonging to $C(0)$. We have the following Lemma.

Lemma 6. *For every $J_k \in C(0)$, $1 \leq k \leq M$, there exists a unique cycle D_k in G_0 satisfying the following properties.*

- (a) *The square J_k is contained in the interior of D_k .*
- (b) *Every edge in the cycle D_k is a boundary edge adjacent to one occupied square of $C(0)$ in the interior and one vacant square in the exterior.*
- (c) *If C is any cycle in G_0 that contains J_k in the interior, then every edge in C either belongs to D_k or is contained in the interior.*

We prove in Theorem 4 that every edge of D_k is also an outermost boundary edge in the graph G_0 . We therefore denote D_k to be the *outermost boundary cycle* containing the square $J_k \in C(0)$.

Proof of Lemma 6: Fix $1 \leq k \leq M$. We first construct a large cycle C_{fin} by merging together all cycles containing the square J_k in their interior. We then show that the cycle C_{fin} satisfies properties (a) – (c).

Construction of the cycle C_{fin} : Let \mathcal{E} be the set of all cycles in the graph G_0 satisfying property (a); i.e., if C is a cycle containing the square J_k in its interior then $C \in \mathcal{E}$. The set \mathcal{E} is not empty since the cycle F_0 formed

by the four edges of J_k belongs to \mathcal{E} . We merge cycles in \mathcal{E} one by one using Theorem 3 to obtain a final cycle C_{fin} .

We set $\mathcal{E}_0 = \mathcal{E}$ and pick a cycle H_0 in $\mathcal{E}_0 \setminus F_0$ using a fixed procedure; for example, using an analogous iterative procedure as described in Section 1 of Ganesan (2014) for choosing paths. We merge cycles F_0 and H_0 and from Theorem 3, we obtain a cycle F_1 in G_0 containing both F_0 and H_0 in its interior. The cycle F_1 also contains the square S_0 in its interior and so satisfies property (a). If it also satisfies property (c), then terminate the procedure and output F_1 .

If the cycle F_1 does not satisfy property (c), then there exists a cycle in $\mathcal{E}'_1 = \mathcal{E}_0 \setminus \{F_0, H_0, F_1\}$ satisfying property (a) but containing at least one edge in the exterior of the cycle F_1 . Let $\mathcal{E}_1 \subset \mathcal{E}'_1$ be the set of all such cycles and pick one such cycle H_1 using a fixed procedure. Merge F_1 and H_1 using Theorem 3 to get a new cycle F_2 .

Repeat the procedure above with the cycle F_2 and this procedure proceeds only for a finite number of steps, since the sets $\{\mathcal{E}_j\}$ form a strictly decreasing sequence of subsets of \mathcal{E} and the set \mathcal{E} has finite number of elements. Let C_{fin} be the final cycle obtained at the end of the procedure. It remains to see that the cycle C_{fin} satisfies properties (a) – (c).

Proof of properties (a) – (c): By construction, the cycle C_{fin} obtained above is unique and satisfies property (a). It also satisfies property (c) because if it did not, the procedure above would not have terminated. It only remains to see that property (b) is true.

Suppose there exists an edge e of the cycle C_{fin} that is not a boundary edge. Since e belongs to the graph G_0 , the edge e is adjacent to an occupied square $A_1 \in \{J_i\}$. But since e is not a boundary edge, the other square $A_2 \in \{J_i\} \setminus A_1$ containing e as an edge is also occupied. One of these squares, say A_1 , is contained in the interior of C_{fin} and the other square A_2 , is contained in the exterior.

The square A_2 and the cycle C_{fin} have the edge e in common and thus more than one vertex in common. We use Theorem 3 to obtain a larger cycle C_{lar} containing both C_{fin} and A_2 in the interior. Since A_2 contains at least one edge in the exterior of C_{fin} , the cycle C_{lar} also contains at least one edge in the exterior of C_{fin} . Moreover, the cycle C_{lar} also contains the occupied square J_k in its interior and therefore satisfies property (a). But since C_{fin} satisfies property (c), this is a contradiction. Thus every edge e of C_{fin} is a boundary edge.

By the same argument above, we also see that the edge e cannot be

adjacent to an occupied square in the exterior of the cycle C_{fin} . Thus e is adjacent to an occupied square in the interior of C_{fin} and a vacant square in the exterior. Thus the cycle C_{fin} satisfies property (b). ■

Proof of Theorem 1: We claim that the set of distinct cycles in the set $\mathcal{D} := \cup_{J_k \in C(0)} \{D_k\}$ obtained in Lemma 6 is the desired outermost boundary ∂_0 and satisfies the properties (i) – (v) mentioned in the statement of the theorem.

The properties (iv) and (v) follow from Lemma 6. To see that (iv) is satisfied, let $J_k \in C(0)$ be any occupied square. The outermost boundary cycle D_k satisfies property (a) of Lemma 6 and so contains the square J_k in its interior. This proves that (iv) is true. To prove (v), let $e \in D_k$ be any edge. Since the cycle D_k satisfies property (b) of Lemma 6, the edge e satisfies (v).

In what follows, we prove (iii), (ii) and (i) in that order.

Proof of (iii): Consider two cycles $D_{k_1} \neq D_{k_2}$. We first see that the cycles D_{k_1} and D_{k_2} have mutually disjoint interiors. We consider various possibilities.

(p1) Every edge in the cycle D_{k_2} is either belongs to or is contained in the interior of the cycle D_{k_1} .

(p2) Every edge in the cycle D_{k_1} is either belongs to or is contained in the interior of the cycle D_{k_2} .

(p3) There are edges $e_1, e_2 \in D_{k_2}$ such that e_1 lies in the interior of cycle D_{k_1} and e_2 lies in the exterior of D_{k_1} .

(p4) There are edges $f_1, f_2 \in D_{k_1}$ such that f_1 lies in the interior of cycle D_{k_2} and f_2 lies in the exterior of D_{k_2} .

If none of the above possibilities hold, then the cycles D_{k_1} and D_{k_2} have mutually disjoint interiors.

To eliminate possibilities (p1) – (p2) we argue as follows. Suppose (p1) holds. The cycle D_{k_1} then contains the square $J_{k_2} \in C(0)$ in its interior and since $D_{k_1} \neq D_{k_2}$, the cycle D_{k_1} also contains an edge lying in the exterior of D_{k_2} . This contradicts the fact that D_{k_2} satisfies property (c) of Lemma 6. This eliminates possibility (p1) and an analogous argument holds for (p2).

We eliminate possibilities (p3) – (p4) as follows. If (p3) holds, then the edge e_2 belongs to a path $P_2 \subset D_{k_2}$ whose every edge lies in the exterior of the cycle D_{k_1} . The path $P_2 \neq D_{k_2}$ since there is at least one edge $e_1 \in D_{k_2}$ lying in the interior of the cycle D_{k_1} . From Theorem 3, we then obtain a cycle E_{12} containing both the cycles D_{k_1} and D_{k_2} in its interior. The cycle E_{12} also contains an edge $e_{12} \in D_{k_2}$ lying in the exterior of the cycle D_{k_1} . Moreover,

the cycle E_{12} contains the occupied square $J_{k_1} \in C(0)$ in its interior and so satisfies property (a) in Lemma 6. But this is a contradiction since the cycle D_{k_1} satisfies property (c) of Lemma 6. This eliminates possibility (p3) and an analogous argument holds for (p4).

We have obtained that D_{k_1} and D_{k_2} have mutually disjoint interiors. If they share more than one vertex in common, we again merge them as described in the previous paragraph and obtain a contradiction. Thus the cycles $D_{k_1} \neq D_{k_2}$ have at most one vertex in common and have mutually disjoint interiors and this proves (iii). ■

Proof of (ii): We use the fact that the graph G_0 is connected. To see this is true, let u_1 and u_2 be vertices in G_0 . Each $u_i, i = 1, 2$ is a corner of an occupied square $S_i \in C(0)$ and by definition, there is a star connected S -path of squares connecting S_1 and S_2 , consisting only of squares in $C(0)$. Thus there exists a path in G_0 from u_1 to u_2 .

To see that $\mathcal{D} = \cup_{S_k \in C(0)} \{D_k\}$ is a connected subgraph of the graph G_0 , we let v_1 and v_2 be vertices in \mathcal{D} that belong to cycles D_{r_1} and D_{r_2} , respectively, for some r_1 and r_2 . If $r_1 = r_2$, then v_1 and v_2 are connected by a path within the cycle D_{r_1} . If $r_1 \neq r_2$, let $P_{12} = (q_1, q_2, \dots, q_{t-1}, q_t)$ be a path of edges in G_0 with endvertices v_1 and v_2 . We iteratively construct a path Q_{12} from P_{12} using only edges of cycles in \mathcal{D} . Every edge q_i in P_{12} is the edge of an occupied square of $C(0)$ and so either belongs to a cycle in \mathcal{D} or is contained in the interior of some cycle in \mathcal{D} . Without loss of generality, we assume that the first edge $q_1 \in P_{12}$ either belongs to the cycle D_{r_1} or is contained in the interior of D_{r_1} .

In the first step of the iteration, we let $s_0 = r_1, i_0 = 1$ and let i_1 be the first time the path P_{12} leaves the cycle D_{s_0} ; i.e., let

$$i_1 = \min\{i \geq i_0 + 1 : q_i \text{ belongs to exterior of } D_{s_0}\}.$$

The edge q_{i_1} has one endvertex w_{i_1} in D_{s_0} and the other endvertex lies in the exterior of D_{s_0} . Let $T_1 \subset D_{s_0}$ be a path consisting only of edges in the cycle D_{s_0} , with endvertices v_1 and w_{i_1} . This completes the first step of the iteration.

For the second step, we use the fact that the edge q_{i_1+1} of the path P_{12} lies in the exterior of the cycle D_{s_0} but belongs to some occupied square of the component $C(0)$. We therefore have that either q_{i_1+1} belongs to some cycle D_{s_1} or is contained in its interior. Also the cycles D_{s_1} and D_{s_0} meet at w_{i_1} and have no other vertex in common.

Repeating the same procedure above, let

$$i_2 = \min\{i \geq i_1 + 1 : q_{i+1} \text{ belongs to exterior of } D_{s_1}\}$$

be the first time P_{12} leaves the cycle D_{s_1} and obtain a path $T_2 \subset D_{s_1}$ with endvertices w_{i_1} and w_{i_2} . As before w_{i_2} is the endvertex of the edge q_{i_2} belonging to the cycle D_{s_1} . We continue the above procedure for a finite number of steps m , until we reach v_2 . By construction, the path T_i obtained at step i , $2 \leq i \leq m$ is connected to $\cup_{1 \leq j \leq i-1} T_j$. The final union of paths $\cup_{1 \leq i \leq m} T_i$ is therefore a connected graph containing only edges in \mathcal{D} and also containing the vertices v_1 and v_2 . This proves (ii). ■

Proof of (i): We first show that every edge in the union of the cycles $\mathcal{D} = \cup_{J_k \in C(0)} \{D_k\}$ is an outermost boundary edge. If e is an edge of a cycle $D_k \in \mathcal{D}$ we have that e is an edge of an occupied square $J_e \in C(0)$ contained in the interior of D_k and is also an edge of a vacant square W_e in the exterior of D_k . If $D_e \in \mathcal{D}$ denotes the outermost boundary cycle containing the square J_e , then from (iii) above we must have that $D_e = D_k$. This is because if $D_e \neq D_k$, then D_e and D_k have mutually disjoint interiors. This cannot happen since both D_e and D_k contain the square J_e in the interior.

If there exists a cycle C in the graph G_0 that contains the edge e in the interior, then both the squares J_e and W_e are contained in the interior of C . Since W_e is exterior to the cycle $D_e = D_k$, the cycle C contains at least one edge in the exterior of D_e . This contradicts the fact that the cycle D_e satisfies property (c) of Lemma 6. Thus e is an outermost boundary edge.

We now argue that no other edge apart from edges of cycles in \mathcal{D} can belong to the outermost boundary. Suppose $e_1 \notin \mathcal{D}$ is an edge of the graph G_0 belonging to some occupied square $A_1 \in C(0)$. From property (iv), the square A_1 is contained in the interior of some cycle $D_k \in \mathcal{D}$. Since $e_1 \notin \mathcal{D}$, the edge e_1 does not belong to D_k and is necessarily contained in the interior of D_k . This proves (i). ■

Circuit C_{out} containing the outermost boundary ∂_0

To obtain the circuit C_{out} , we first compute the cycle graph H_{cyc} as follows. Let E_1, E_2, \dots, E_n be the distinct outermost boundary cycles in \mathcal{D} . Represent E_i by a vertex i in H_{cyc} . If E_i and E_j share a corner, we draw an edge $e(i, j)$ between i and j . We have the following properties regarding the

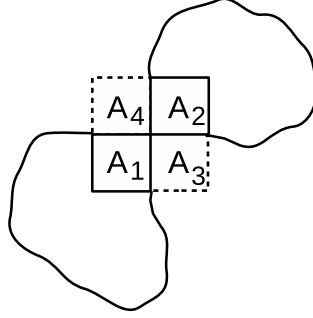


Figure 4: Only two cycles of the outermost boundary can meet at a single point.

graph H_{cyc} .

(y1) Let $P = (e(i_1, i_2), e(i_2, i_3), \dots, e(i_{m-1}, i_m))$ be a path of edges in the graph H_{cyc} , where $i_j, 1 \leq j \leq m$ are vertices in H_{cyc} . Let $u \in E_{i_1}$ and $v \in E_{i_m}$ be any two vertices. There is a path P_{uv} with endvertices u and v and consisting only of edges of the cycles $\{E_{i_k}\}_{1 \leq k \leq m}$.

(y2) The graph H_{cyc} is acyclic and connected and we record the following for future use.

We have that the cycle graph H_{cyc} is a tree. (3.1)

Proof of (y1) – (y2): To obtain the path P_{uv} , we proceed as follows. Set $w_1 = u, w_{m+1} = v$ and for $2 \leq j \leq m$, let w_j denote the vertex common the cycles $E_{i_{j-1}}$ and E_{i_j} . For $1 \leq j \leq m$, the vertices w_j and w_{j+1} both belong to the cycle E_{i_j} and so there is a path Q_j with endvertices w_j and w_{j+1} consisting only of edges of the cycle E_{i_j} . Since the cycles $\{E_i\}$ are edge disjoint (property (iii)), the paths $\{Q_j\}_{1 \leq j \leq m}$ are edge disjoint. Therefore the union of the paths $\cup_{j=1}^m Q_j$ is a path with endvertices w_1 and w_{m+1} and containing only edges in the cycles $\{E_{i_j}\}_{1 \leq j \leq m}$.

To prove (y2), we use property (ii) of the outermost boundary to obtain that the union of the cycles $\cup_{1 \leq i \leq n} E_i$ is a connected graph. Therefore, the graph H_{cyc} is connected. We use property (y1) to prove that H_{cyc} is acyclic.

Suppose H_{cyc} contains a cycle $C = (e(r_1, r_2), \dots, e(r_s, r_1))$. Again $1 \leq r_i \leq n, i = 1, 2, \dots, s$ are vertices of the graph H_{cyc} . Using property (iii) of the outermost boundary, we have that the cycles $E_{r_1} = (u_1, u_2, \dots, u_m, u_1)$ and E_{r_2} have exactly one vertex, say u_{j_2} , in common. Similarly, the cycles E_{r_1}

and E_{r_s} have exactly one vertex, u_{j_s} , in common.

We have that the indices $j_2 \neq j_s$ since three boundary cycles cannot meet at a point. We assume that some cycle $E_j, j \neq r_1, r_2$ also contains the vertex u_{j_2} and obtain a contradiction as follows. All the four edges $g_i, 1 \leq i \leq 4$, containing u_{j_2} as an endvertex belongs to either E_{r_1} or E_{r_2} . The illustrated in Figure 4 where the occupied square $A_1 \in \{S_k\}$ is in the interior of the cycle E_{r_1} and the occupied square $A_2 \in \{S_k\}$ is in the interior of E_{r_2} . The vertex u_{j_2} is the vertex common to all the four squares $\{A_i\}_{1 \leq i \leq 4}$. Therefore if the cycle E_j contains u_{j_2} as an endvertex, then E_j contains at least one of the edges $g_i, 1 \leq i \leq 4$. But this contradicts the property (iii) that the cycles in the outermost boundary have at most one vertex in common.

Let Q_1 and R_1 be the two subpaths of E_{r_1} with endvertices u_{j_2} and u_{j_s} so that $Q_1 \cup R_1 = E_{r_1}$. The vertex $u_{j_2} \in E_{r_2}$ and $u_{j_s} \in E_{r_s}$ and so by property (y1) above, there exists a path P_{2s} with endvertices u_{j_2} and u_{j_s} , consisting only of edges in $\{E_{r_i}\}_{2 \leq i \leq s}$. From the property (iii) of the outermost boundary, we have that the cycles $\{E_i\}$ have mutually disjoint interiors. Therefore every edge in the cycle $E_{r_i}, 2 \leq i \leq s$ lies in the exterior of the cycle E_{r_1} . In particular, every edge in the path P_{2s} lies in the exterior of the cycle E_{r_1} .

In Figure 3, we illustrate the cycle E_{r_1} as $XUZVX$ and the paths R_1 and Q_1 are respectively denoted by the segments XVZ and XUZ . The path P_{2s} lying in the exterior of the cycle E_{r_1} is the segment XYZ . One of the cycles $P_{2s} \cup Q_1$ or $P_{2s} \cup R_1$ contains the cycle E_{r_1} in its interior. We call this cycle C_{12} . In Figure 3, the cycle $C_{12} = P_{2s} \cup R_1$ is represented by $XYZVX$ and contains the cycle $E_{r_1} = XUZVX$ in its interior.

Let $J_k \in \{S_j\}$ be any occupied square of the component $C(0)$ in the interior of the cycle E_{r_1} . We have from Lemma 6 that the cycle $E_{r_1} = D_k$, the outermost boundary cycle containing the square J_k and satisfies properties (a), (b) and (c) mentioned in the statement of Lemma 6. The cycle C_{12} also contains J_k in its interior and thus satisfies property (a). Moreover, it contains at least one edge in the exterior of the cycle E_{r_1} contradicting the fact that E_{r_1} satisfies property (c). Thus the graph H_{cyc} is acyclic. This proves (y2). \blacksquare

Using (3.1), we obtain the desired circuit for the outermost boundary ∂_0 , iteratively, by considering an increasing sequence of tree subgraphs of the tree H_{cyc} . The vertex set of H_{cyc} is $\{1, 2, \dots, n\}$ and for vertex $v \in \{1, 2, \dots, n\}$, let $\mathcal{N}(v)$ be the neighbours of v in the tree H_{cyc} . Set $q_1 = 1$, $H_1 = \{q_1\}$ and

for $i \geq 1$, let $V(H_i)$ is the vertex set of the graph H_i . For $i \geq 1$, we have the following properties regarding the graph H_i .

- (z1) We have that H_i is a tree subgraph of H_{cyc} with $V(H_i) = \{q_1, \dots, q_i\}$. If $i \leq n-1$, the following additional condition holds.
- (z2) There exists a vertex $q_{i+1} \notin V(H_i)$ that is adjacent to exactly one vertex $v_{i+1} \in V(H_i)$. Pick the least such q_{i+1} and set $V(H_{i+1}) = V(H_i) \cup \{q_{i+1}\}$.

Proof of (z1) – (z2) for $i = 1$: The proof of (z1) is true by construction. To see (z2) is true, we argue as follows. The vertex set $V(H_i)$ of the graph H_i satisfies $\#V(H_i) \leq n-1$ and so there is at least one vertex in $\{1, 2, \dots, n\} \setminus H_i$. We recall that $\mathcal{N}(v)$ denotes the neighbours of the vertex v in the graph H_{cyc} . If $\mathcal{N}(v) \subset H_i$ for all $v \in H_i$, then the graph H_i is a (connected) component of the graph H_{cyc} with $\#V(H_i) \leq n-1$. This means that H_{cyc} is not connected, a contradiction since H_{cyc} is a tree, (see property (y2) above). Thus there exists at least one vertex $v_{i+1} \in H_i$ containing a neighbour in $\{1, 2, \dots, n\} \setminus V(H_i)$. Pick the least such v_{i+1} and let q_{i+1} be the least indexed neighbour of v_{i+1} in $\{1, 2, \dots, n\} \setminus V(H_i)$.

Suppose now that there is another vertex $w_{i+1} \in H_i, w_{i+1} \neq v_{i+1}$ that is also adjacent to $q_{i+1} \notin H_i$. The graph H_i is a tree and so the vertices w_{i+1} and v_{i+1} are connected by a path P_{vw} consisting only of edges in H_i . Let f be the edge between q_{i+1} and v_{i+1} in the tree H_{cyc} and let g be the edge between the vertices q_{i+1} and w_{i+1} in the tree H_{cyc} . The union $P \cup \{f, g\}$ contains a cycle consisting of edges in H_{cyc} , a contradiction to (3.1). ■

The property (z2) is used to proceed to the next step of the iteration and the graph H_2 obtained at the end of the second iteration again satisfies properties (z1) – (z2). The above procedure continues for n steps and the final graph $H_n = H_{cyc}$.

We use the graphs $H_i, 1 \leq i \leq n$ to construct the desired circuit for the outermost boundary iteratively. We recall that $\{E_i\}_{1 \leq i \leq n}$ are the cycles in the outermost boundary ∂_0 . The graph H_1 contains a single vertex $\{1\}$ and set $\Pi_1 = E_1$ to be the circuit obtained at the end of the first iteration. For $1 \leq i \leq n$, let Π_i be the circuit obtained at the end of the i^{th} iteration. We have the following properties.

- (x1) The circuit Π_i contains all the edges belonging to the cycles $E_v, v \in V(H_i) = \{q_1, \dots, q_i\}$.
- (x2) If $i \leq n-1$, then the circuit Π_i does not contain any edge from $\bigcup_{j=i+1}^n E_{q_j}$ and shares exactly one vertex with the cycle $E_{q_{i+1}}$.

Proof of (x1)–(x2) for $i = 1$: The proof of (x1) is true by construction. To see (x2) is true, we use the fact that tree H_i has vertex set $\{q_1, q_2, \dots, q_i\}$. From property (z2) above, we have that the vertex q_{i+1} is adjacent to exactly one vertex $v_{i+1} \in V(H_i)$. The corresponding cycle $E_{q_{i+1}}$ therefore shares exactly one vertex with the cycle $E_{v_{i+1}} \subset \Pi_i$ and does not share a vertex with any other cycle $E_v, v \in H_i$. ■

We use property (x2) to proceed to the next step of the iteration. Let $\Pi_i = (c_1, c_2, \dots, c_r)$ be the edges of the circuit Π_i traversed in that order and let Π_i meet the cycle $E_{q_{i+1}} = (d_1, d_2, \dots, d_l)$ at some vertex a common to the edges c_1 and d_l . We then form the new circuit $\Pi_{i+1} = (d_1, d_2, \dots, d_l, c_1, c_2, \dots, c_r, c_1)$. The circuit Π_{i+1} also satisfies properties (x1) – (x2) and continuing this way iteratively, the final circuit Π_n contains the edges of all the cycles $E_v, 1 \leq v \leq n$. The circuit Π_n is therefore the desired circuit of the outermost boundary ∂_0 . ■

Proof of Theorem 2: Let D_0 be the outermost boundary cycle containing the square S_0 as in Lemma 6. It satisfies the conditions (i) and (ii) in the statement of the theorem and is unique and thus $\partial_0^+ = D_0$. *qed*

4 Proof of Theorem 4

We first see that an open left right crossing and a closed dual top bottom crossing cannot occur simultaneously. Let P_1 be an open left right crossing of R . If there exists a closed top bottom dual crossing P_2 of R , then the paths P_1 and P_2 intersect and in particular, there is some edge $e \in P_1$ which is open but its dual edge $f(e) \in P_2$ is closed. This is a contradiction.

We henceforth assume that R does not contain an open left right crossing. Let \mathcal{C} be the set of all vertices in R that are connected to some vertex in R_{left} by an open path of edges. We recall that R_{left} denotes the left edge of the rectangle R and every vertex $(0, i) \in R_{left}$ belongs to \mathcal{C} . Also since R has no open left right crossing, no vertex in \mathcal{C} belongs to the right edge R_{right} of R .

We also recall that every vertex in the rectangle R is the centre of some square in $\{S_k\}$ contained in the dual graph G . Let $J_i \subset \{S_k\}$ be the square with centre as $(0, i) \in R_{left}$. For $z \in R \setminus R_{left}$, let $J_z \in \{S_k\}$ be the square with centre z . We say that J_z is occupied if the corresponding vertex $z \in \mathcal{C}$. Else we set J_z to be vacant. Let C_{left} be the plus connected component containing

the square $J_0 = S_0$ with centre as the origin. Every square $J_i, 1 \leq i \leq n$ belongs to C_{left} . From Theorem 2, we have that the outermost boundary ∂_{left}^+ of C_{left} is a single cycle.

We extract a closed dual top bottom crossing as a subpath of ∂_{left}^+ . For $0 \leq i \leq n$, let $h_i(t), h_i(r), h_i(b)$ and $h_i(l)$ denote the top, right, bottom and left edges of the square J_i . Also let R_{top} and R_{bottom} be the top and bottom edges of the rectangle R . We enumerate the following properties of ∂_{left}^+ needed for future use.

(x1) The path $\Pi_1 = (h_n(t), h_n(l), h_{n-1}(l), \dots, h_0(l), h_0(b))$ is a subpath of ∂_{left}^+ . Let $\Pi_2 := \partial_{left}^+ \setminus \Pi_1 = (f_1, \dots, f_r)$ where the edge f_1 shares an endvertex with the top edge $h_n(t)$ of Π_1 and the edge f_r shares an endvertex with bottom edge $h_0(b)$ of Π_1 .

(x2) If (x, y) is an endvertex of some edge in Π_2 , then $\frac{1}{2} \leq x \leq m - \frac{1}{2}$ and $-\frac{1}{2} \leq y \leq n + \frac{1}{2}$. In particular, no edge in subpath Π_2 intersects the left edge R_{left} of R and no edge in Π_2 intersects the right edge R_{right} of R .

(x3) Let $\Pi = (h_1, \dots, h_s)$ be any subpath of Π_2 with one endvertex of h_1 lying above R_{top} and one endvertex of h_s lying below R_{bottom} . There are indices $1 \leq k_1 < k_2 \leq s$ such that h_{k_1} intersects the top edge of R and the edge h_{k_2} intersects the bottom edge of R .

Essentially property (x3) implies that the path Π_2 intersects the top edge of R before intersecting the bottom edge.

Proof of (x1) – (x3): We prove (x1) as follows. The squares $J_i, 1 \leq i \leq n$ are the left most squares in the plus connected component C_{left} in the sense that the square J_i has centre $(0, i)$ in the left edge R_{left} of the rectangle R . Suppose the left edge $h_i(l)$ of J_i does not belong to the outermost boundary cycle ∂_{left}^+ . Some edge e in the cycle ∂_{left}^+ then intersects the line $y = i$ at (x, i) , where $x \leq -\frac{3}{2}$. This is true because the edge $h_i(l)$ is contained in the line $x = -\frac{1}{2}$ and every edge of ∂_{left}^+ intersects the line $y = i$ at $(\frac{k}{2}, i)$ for integer $k \neq 0$. The edge e belongs to a square J_v whose vertex $v \in \mathcal{C}$. The vertex v has x -coordinate at most -1 , a contradiction since every vertex in \mathcal{C} belongs to R .

An analogous argument as above using top most and bottom most squares obtains that $h_n(t)$ and $h_0(b)$ belong to ∂_{left}^+ . Since every vertex apart from the endvertices have degree two in Π_1 , the path Π_1 is a subpath of the cycle ∂_{left}^+ .

We prove (x2) is true as follows. The bounds for y -coordinates are true since we only consider edges of squares with centre in the rectangle R . Since

the subpath $\Pi_1 \subset \partial_{left}^+$, we have that no edge in Π_2 intersects the left edge of R . Because if there exists an edge f intersecting the left edge of R , then f shares an endvertex v with the left edge $h_i(l)$ of some square J_i , $0 \leq i \leq n-1$. Since the edges $h_i(l)$ and $h_{i+1}(l)$ both belong to the cycle ∂_{left}^+ , the vertex v then has degree three in ∂_{left}^+ , a contradiction. Thus if (x_1, y_1) is the endvertex of some edge in Π_2 , we have $x_1 \geq \frac{1}{2}$.

Suppose now that some edge g in Π_2 has endvertex intersecting the line $x = m + \frac{1}{2}$. The edge g is the edge of a square J_v whose centre v lies in the right edge R_{right} of R and also belongs to the cluster \mathcal{C} . This means that \mathcal{C} contains a left right crossing of R , a contradiction. Thus the x -coordinate of the endvertex (x_1, y_1) of any edge in Π_2 satisfies $x_1 \leq m - \frac{1}{2}$.

To prove (x3), we use the fact that the path Π crosses both the lines $y = n$ and $y = 0$. Moreover, it crosses $y = n$ before crossing $y = 0$; i.e., there are edges h_{k_1} and h_{k_2} such that $k_1 < k_2$ and h_{k_1} crosses the top line $y = n$ and the edge h_{k_2} crosses the bottom line $y = 0$. From property (x2), we have that the endvertices of h_{k_1} lie in the cylinder $\frac{1}{2} \leq x \leq m - \frac{1}{2}$ and so the edge h_{k_1} intersects the top edge of R and similarly, the edge h_{k_2} intersects the bottom edge of R . ■

We use the above properties to extract the necessary dual top bottom crossing. Let k_2 be the least index such that the edge $f_{k_2} \in \Pi_2$ crosses the line $y = 0$. From property (x2), we have that one endvertex v_{k_2} of f_{k_2} lies in the interior of R . The other endvertex u_{k_2} lies in the exterior of R . Also, either f_{k_2-1} or f_{k_2+1} contain the endvertex v_{k_2} . If f_{k_2+1} contains the endvertex v_{k_2} , then the edge f_{k_2-1} contains u_{k_2} as an endvertex and u_{k_2} lies below the bottom edge R_{bottom} of R . From property (x3), we then have that the subpath (f_1, \dots, f_{k_2-1}) contains an edge that crosses the line $y = 0$, a contradiction to the definition of k_2 .

From the discussion in the previous paragraph, we have that the edge f_{k_2-1} contains v_{k_2} as an endvertex and since the height n of the rectangle R is at least two, the edge f_{k_2-1} does not intersect the top edge of R . Therefore from property (x2), both the endvertices of the edge f_{k_2} lie in the interior of R . Since an endvertex the first edge f_1 lies above $y = n$, the subpath (f_1, \dots, f_{k_2-1}) crosses the line $y = n$ and so there exists an index $k_1 < k_2 - 1$ such that the edge f_{k_1} crosses the top line $y = n$. Let r_1 be the largest index less than $k_2 - 1$ such that f_{r_1} crosses the top edge R_{top} of R . Let v_{r_1} be the endvertex of f_{r_1} belonging in the interior of R . Arguing as in the previous

paragraph, we have that the edge f_{r_1+1} contains v_{r_1} as an endvertex and both the endvertices of f_{r_1+1} lie in the interior of R .

Every edge in the subpath $\Pi_t = (f_{r_1+1}, \dots, f_{k_2-1})$ has both its endvertices in the interior of the rectangle R . Therefore the union $(f_{r_1}, \Pi_t, f_{k_2})$ is a dual top bottom crossing of R . ■

5 Outermost boundary in oriented percolation

We recall that R_{left} denotes the left edge of the rectangle R . For simplicity, we denote the square S_z'' containing the vertex $z \in \mathbb{Z}^2$ as the centre, simply as S_z . The edges in S_z are called dual edges and in this subsection, we do not consider orientation in the dual edges. We introduce the corresponding orientation in the next subsection.

Let $f_{NE}(z)$ denote the unoriented (dual) edge of the square S_z with endvertices $(i-1, j)$ and $(i, j+1)$ and let $f_{SE}(z)$ denote the unoriented edge with endvertices $(i, j+1)$ and $(i+1, j)$ with notations NE and SE standing for north east and south east, respectively. Similarly we define north west unoriented edge $f_{NW}(z)$ as the edge with endvertices $(i+1, j)$ and $(i, j-1)$ and the south west unoriented edge $f_{SW}(z)$ with endvertices $(i, j-1)$ and $(i-1, j)$.

For illustration we refer to Figure 2(b) above, where $F_1F_2F_3F_4$ represents the square S_A containing the point $A = (0, 1) \in R_{left}$ as the centre. The segments F_1F_2, F_2F_3, F_3F_4 and F_4F_1 respectively denote the dual edges $f_{SE}(0, 1), f_{SW}(0, 1), f_{NW}(0, 1)$ and $f_{NE}(0, 1)$.

Let \mathcal{C} denote the collection of all vertices in the open oriented cluster defined as follows. If $x = (0, j) \in R_{left}$ then $x \in \mathcal{C}$ if and only if j is odd. Since the height n of the rectangle is at least one, we have that $(0, 1) \in \mathcal{C}$ and so $\mathcal{C} \neq \emptyset$. If $x \in R \setminus R_{left}$, then $x \in \mathcal{C}$ if and only if there is an open oriented path P_x with one endvertex in $\mathcal{C} \cap R_{left}$ and the other endvertex as x . We recall that every open oriented path contains one arrow endvertex and one non arrow endvertex and by construction, the non arrow endvertex of P_x is some $(0, j) \in \mathcal{C} \cap R_{left}$ and the arrow endvertex is x .

For $z \in \mathbb{Z}^2$, we say the square S_z is *occupied* if either $z \in \mathcal{C}$ and *vacant*

otherwise. The resulting union $\mathcal{Q} = \cup_{z \in \mathcal{C}} S_z$ of occupied squares is a star connected component. From Theorem 1, the outermost boundary of \mathcal{Q} is a unique connected union of cycles $\cup_{i=1}^h C_i$ consisting of dual edges in $\cup_{z \in \mathcal{C}} S_z$ and satisfying the following properties:

- (a1) Every vertex $z \in \mathcal{C}$ is in the interior of some cycle C_i .
- (a2) For any $1 \leq i \leq h$, every edge in the cycle C_i is a boundary edge adjacent to one occupied square contained in the interior of C_i and one vacant square in the exterior.
- (a3) The cycles $\{C_i\}$ have mutually disjoint interiors and for $i \neq j$, the cycles C_i and C_j intersect at most at one point.

The following is the main result we prove in this Section.

Theorem 7. *With orientation as introduced in Section 1, every cycle $C_i, 1 \leq i \leq h$, in the outermost boundary ∂_0 is an oriented dual cycle; i.e. an oriented dual path with coincident endvertices.*

We also derive auxiliary properties along the way used in obtaining the dual crossing in the next section.

In the following two subsections, we consider the outermost boundary without orientation and in the final subsection, we introduce orientation and prove Theorem 7.

Contiguous block property of the outermost boundary

In the main result of this subsection, we state and prove the contiguous block property for the outermost boundary. We recall that R_{left} is the left edge of the rectangle R .

- (b1) If $z \in \mathcal{C} \cap R_{left}$, then the dual edges $f_{NW}(z)$ and $f_{NE}(z)$ belonging to the square S_z are consecutive edges in some cycle C_i of the outermost boundary. Also, both $f_{NW}(z)$ and $f_{NE}(z)$ lie in the exterior of every cycle $C_j, 1 \leq j \leq h, j \neq i$.
- (b2) If vertices $(0, j_1), (0, j_2) \in \mathcal{C} \cap R_{left}, j_1 < j_2$ both belong to the interior of some cycle C_i of the outermost boundary, then every $(0, j) \in \mathcal{C} \cap R_{left}$ with $j_1 \leq j \leq j_2$ belongs to the interior of C_i .

The property (b2) is the contiguous block property which says that the set of vertices of R_{left} lying in the interior of a boundary cycle C_i forms a contiguous block.

Proof of (b1) – (b2): We prove (b1) – (b2) for the cycle C_1 containing the vertex $(0, 1) \in \mathcal{C} \cap R_{left}$ in its interior. Let $z = (0, j) \in \mathcal{C} \cap R_{left}$ be any vertex in the interior of the cycle C_1 of the outermost boundary. We prove the property for $f_{NW}(z)$ and an analogous proof holds for $f_{SW}(z)$. If the edge $f_{NW}(z)$ does not belong to the cycle C_1 , then it lies in the interior of C_1 and so both the squares containing $f_{NW}(z)$ as an edge lie in the interior of C_1 . Since the squares S_z and S_{z_1} with centre $z_1 = (-1, j - 1)$ both contain $f_{NW}(z)$ as an edge, the vertex z_1 lies in the interior of the cycle C_1 . This is a contradiction since every vertex in the interior of C_1 either lies in the interior of the rectangle R or belongs to the boundary and the vertex z_1 lies in the exterior of R . This proves (b1).

We assume that (b2) is not true and arrive at a contradiction. Suppose there are integers $j_1 < j_2$ such that the following two statements (a) – (b) hold. (a) The vertices $(0, j_1), (0, j_2) \in \mathcal{C} \cap R_{left}$ both belong to the interior of the cycle $C_1 = (f_1, f_2, \dots, f_t)$ of the outermost boundary. (b) Every intermediate vertex $(0, j) \in \mathcal{C} \cap R_{left}, j_1 + 1 \leq j \leq j_2 - 1$, belongs to the exterior of C_1 .

We recall that R_{left} is the left edge of the rectangle R and \mathcal{C} is the set of vertices in the rectangle R which are connected by an oriented open path to some vertex in $\mathcal{C} \cap R_{left} = \{(0, j), 1 \leq j \leq n, j \text{ odd}\}$. We therefore assume that $j_1 + 2 < j_2$ and arrive at a contradiction. Since the vertex $(0, j_1) \in \mathcal{C} \cap R_{left}$ lies in the interior of the cycle C_1 , we have from property (b1) that the NE edge $f_{NE}(0, j_1)$ associated with the vertex $(0, j_1)$ belongs to C_1 and $f_{i_1} = f_{NE}(0, j_1)$ for some index $1 \leq i_1 \leq t$. The edge f_{i_1} contains $(0, j_1 + 1)$ as an endvertex.

In Figure 5, we illustrate the scenario in the previous paragraph. The points $A = (0, 0), A_2 = (0, 2)$ and the mid point of AA_2 is $(0, 1)$. The square $S_{0,1}$ with centre $(0, 1)$ is denoted by $AA_1A_2A_3$. The cycle C_1 is denoted by the sequence $AA_1A_2B_1BWCC_1C_2D_1DXA_3A$. The point $B = (0, 4)$ denotes the vertex $(0, j_1 + 1)$ and the edge f_{i_1} is denoted by the segment BB_1 .

In an analogous manner, the north west edge $f_{NW}(0, j_2)$ of the vertex $(0, j_2) \in \mathcal{C} \cap R_{left}$ also belongs to the cycle C_1 and moreover $f_{NW}(0, j_2) = f_{i_2}$ for some index $1 \leq i_2 \leq r$. The edge f_{i_2} contains $(0, j_2 - 1)$ as an endvertex. In Figure 5, the point C denotes the vertex $(0, j_2 - 1)$ and the edge f_{i_2} is denoted by the segment CC_1 .

Let P_{12} be the path containing the union of the edges $\{f_{NW}(0, j), f_{NE}(0, j)\}, j_1 + 1 \leq j \leq j_2 - 1$. From property (b1), we have that

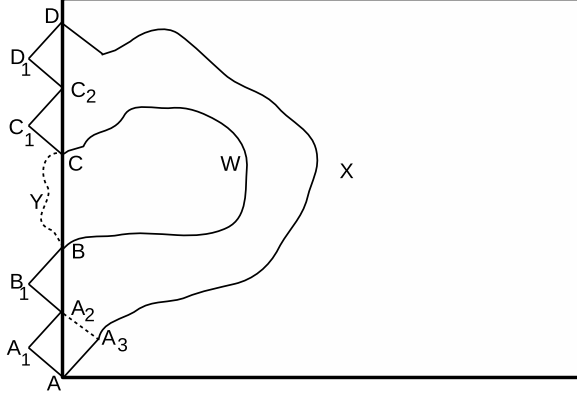


Figure 5: Illustration of the square $S_{0,1} = AA_1A_2A_3$ containing the point $(0,1) \in \mathcal{C} \cap R_{left}$. Also depicted is the cycle C_1 containing the square $S_{0,1}$ in its interior.

every edge in P_{12} lies in the exterior of the cycle C_1 and the path P_{12} contains $(0, j_1 + 1)$ and $(0, j_2 - 1)$ as endvertices. The dotted line BYC denotes P_{12} in Figure 5.

The cycle $C_1 = Q_1 \cup R_1$ is the union of two paths with endvertices $(0, j_1 + 1)$ and $(0, j_2 - 1)$. The union of the paths $P_{12} \cup Q_1$ and $P_{12} \cup R_1$ are therefore two cycles with the following property. Exactly one of the cycles, say $P_{12} \cup R_1$ contains the cycle C_1 in its interior and the other cycle $P_{12} \cup Q_1$ has mutually disjoint interior with the cycle C_1 . In Figure 5, the segment BWC denotes Q_1 and the cycles $P_{12} \cup BWC$ and C_1 have mutually disjoint interiors.

The cycle $P_{12} \cup R_1$ contains the cycle C_1 in its interior and at least one edge in the exterior of C_1 . This contradicts the construction of the cycle C_1 , which is the outermost boundary cycle containing the occupied square $S_{0,1}$; see condition (c) of Lemma 6. ■

Definition and properties of the integers $\{m_i\}$

To study the dual edges with orientation, we need a couple of additional properties regarding the outermost boundary cycles. We use contiguous block property (b2) and define an increasing sequence of integers

$$1 = m_1 < m_2 < \dots < m_h < m_{h+1} \quad (5.2)$$

as follows. We assume that the cycle C_1 contains the vertex $(0, 1) \in \mathcal{C} \cap R_{left}$ in its interior. We recall that R_{left} is the left edge of the rectangle R and $\mathcal{C} \cap R_{left} = \{(0, j) : 1 \leq j \leq n, j \text{ odd}\}$. Let $m_2 - 2$ be the largest index j such that $(0, j)$ lies in the interior of cycle C_1 . The cycle C_1 contains vertices $(0, m_1)$ and $(0, m_2 - 2)$ in its interior and from the contiguous block property (b2), we have that C_1 contains all the vertices $(0, j) \in \mathcal{C} \cap R_{left}, m_1 \leq j \leq m_2 - 2$ in its interior and no other vertices of R_{left} in its interior.

Since the vertex $(0, m_2 - 2)$ lies in the interior of the cycle C_1 , we have that m_2 is odd and so the smallest integer j such that $(0, j)$ lies in the interior of some cycle $C_j, 2 \leq j \leq h$ is $j = m_2$. We assume that the vertex $(0, m_2)$ lies in the cycle C_2 and proceeding as in the previous paragraph, we obtain an index m_3 such that all vertices $(0, j) \in \mathcal{C} \cap R_{left}, m_2 \leq j \leq m_3 - 2$ lie in the interior of the cycle C_2 . Also every other vertex in $\mathcal{C} \cap R_{left}$ lies in the exterior of C_2 .

Continuing this way iteratively, we obtain the sequence $\{m_i\}$. The final integer m_{h+1} satisfies the following bounds

$$n + 1 \leq m_{h+1} \leq n + 2 \quad (5.3)$$

where we recall that n is the height of the rectangle R .

Proof of (5.3): To obtain the second inequality in (5.3), we argue as follows. The final cycle C_h contains all the vertices $\{(0, j) \in \mathcal{C} \cap R_{left}, m_h \leq j \leq m_{h+1} - 2\}$ in its interior. The top most vertex in R_{left} is $(0, n)$ and so $m_{h+1} - 2 \leq n$.

To obtain the lower bound in (5.3), we assume $m_{h+1} \leq n$ and arrive at a contradiction. Since $m_{h+1} - 2 \leq n - 2$, the vertices $(0, n - 1)$ and $(0, n)$ both lie in the exterior of all the cycles $C_i, 1 \leq i \leq h$. But one of the integers $n - 1$ or n is odd and so one of the vertices $(0, n - 1)$ or $(0, n)$ belongs to $\mathcal{C} \cap R_{left}$. Also, by construction, every vertex in \mathcal{C} lies in the interior of some cycle $\{C_i\}$ (see property (a1)). This leads to a contradiction. \blacksquare

We have the following properties regarding the integers $\{m_i\}$. For each property, we also provide the corresponding illustration in Figure 6.

(d1) Fix $1 \leq i \leq h$. The path

$$\begin{aligned} \Gamma_i = & (f_{NW}(0, m_i), f_{NE}(0, m_i), f_{NW}(0, m_i + 1), f_{NE}(0, m_i + 1), \\ & \dots, f_{NW}(0, m_{i+1} - 2), f_{NE}(0, m_{i+1} - 2)). \end{aligned}$$

is a subpath of the cycle C_i with endvertices $(0, m_i - 1)$ and $(0, m_{i+1} - 1)$. The path $\Delta_i = C_i \setminus \Gamma_i$ also has endvertices $(0, m_i - 1)$ and $(0, m_{i+1} - 1)$. Let

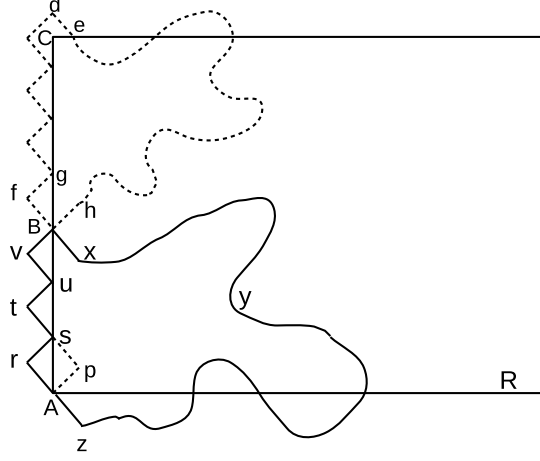


Figure 6: The cycles C_1 (thick line) and C_2 (dotted line) are shown meeting at the point B on the left edge of the rectangle R .

$e_i \in \Delta_i$ be the edge sharing an endvertex with the first edge $f_{NW}(0, m_i) \in \Gamma_i$ and let $g_i \in \Delta_i$ be the edge sharing an endvertex with the last edge $f_{NE}(0, m_{i+1} - 2) \in \Gamma_i$. Every edge in $\Delta_i \setminus \{e_i, g_i\}$ has both endvertices in $R \setminus R_{left}$. Every edge in Δ_i has both its endvertices to the right of R_{left} .

The property (d1) is illustrated in Figure 6 for the case when there are two cycles C_1 and C_2 in the outermost boundary. The cycle C_1 is drawn with thick lines and the cycle C_2 is drawn with dotted lines. The vertices $(0, 0)$, $(0, 1)$, $(0, 2)$ etc are respectively denoted by the points A , the mid point of the segment As , the point s etc. The segment Ar represents the first edge $f_{NW}(0, 1) \in C_1$ containing the origin. The path $ArstuvB$ is the subpath $\Gamma_1 \subset C_1$. The path Δ_1 is the wavy segment $BxyzA$. The line segments Bx and Az respectively denote the edges g_1 and e_1 .

(d2) Suppose that $h \geq 2$. For $1 \leq i \leq h - 1$, the cycles C_i and C_{i+1} intersect only at the vertex $(0, m_{i+1} - 1) \in R_{left}$ and have no other vertex in common. For $2 \leq i \leq h$, the edge $e_i = f_{SW}(0, m_i)$ and for $1 \leq i \leq h - 1$, the edge $g_i = f_{SE}(0, m_{i+1} - 2)$.

In Figure 6, the cycles C_1 and C_2 meet at the point $B = (0, m_2 - 1) = (0, 6)$ on the left edge R_{left} of the rectangle R . The line segments Bx and Az respectively denote the edges $g_1 = f_{SE}(0, m_2 - 2) \in \Delta_1$ and $e_1 \in \Delta_1$. From

the figure, we also see that the segment Ap representing the edge $f_{SW}(0, 1)$ need not necessarily belong to the first cycle C_1 . Thus the second statement of property (d2) regarding the edge e_i need not hold for $i = 1$. An analogous argument holds for the edge g_h belonging to the top most cycle C_h .

The edges $f_{NW}(0, m_2 - 2)$ and $e_2 = f_{SW}(0, m_2 - 2)$ belonging to the cycle C_2 are represented by the segments Bf and Bh . The corner C of the rectangle R belongs to $\mathcal{C} \cap R_{left}$ and so represents the vertex $(0, m_3 - 2) = (0, 13)$. The edge $g_2 = f_{SE}(0, m_3 - 2)$ represented by the segment de lies in the exterior of the rectangle R .

(d3) Suppose $h \geq 3$. The cycle C_1 intersects only the cycle C_2 and no other cycle $C_j, 2 \leq j \leq h$. The cycle C_h intersects only the cycle C_{h-1} and no other cycle $C_j, 1 \leq j \leq h - 2$. For every $2 \leq i \leq h - 1$, the cycle C_i intersects only the cycles C_{i-1} and C_{i+1} and no other cycle in the outermost boundary.

The properties (d1) – (d3) are also used in the next subsection to obtain properties regarding the boundary cycles $\{C_i\}$ with orientation.

Proof of (d1) – (d3): We prove the statement of (d1) regarding the subpath Γ_i for $i = 1$. An analogous proof holds for general i . By definition of the integers $\{m_i\}$, the vertices $(0, j) \in \mathcal{C} \cap R_{left}, 1 \leq j \leq m_2 - 2$ lie in the interior of the cycle C_1 . We recall that $\mathcal{C} \cap R_{left} = \{(0, j) \in R_{left} : j \text{ odd}\}$. From property (b1), we therefore have that the edges $f_{NW}(0, j)$ and $f_{NE}(0, j), m_1 = 1 \leq j \leq m_2 - 2, j \text{ odd}$, are consecutive edges of the cycle C_1 . The edge $f_{NE}(0, j)$ shares the endvertex $(0, j + 1)$ with the edge $f_{NW}(0, j + 2)$ for all j and so Γ_1 is an subpath of the cycle C_1 .

It remains to see that every edge in $\Delta_1 \setminus \{e_1, g_1\}$ has both endvertices in $R \setminus R_{left}$. The edge e_1 contains $(0, m_1 - 1)$ as an endvertex and the edge g_1 contains $(0, m_2 - 1)$ as an endvertex. Also, every vertex $(0, j), 0 = m_1 - 1 \leq j \leq m_2 - 1$ either lies in the interior of the cycle C_1 or belongs to the path Γ_1 . Therefore if there exists an edge $e \in Q_1$ intersecting the left edge R_{left} at some point $(0, y)$, then $y \geq m_2$ or $y \leq m_1 - 2$.

If $y \geq m_2$, then e is the edge of some square S_{0, y_1} with centre $(0, y_1)$, where $y_1 \geq m_2$. But this implies that y_1 is odd and so $(0, y_1) \in \mathcal{C} \cap R_{left}$. Thus the square S_{0, y_1} is occupied and if it does not lie in the interior of the cycle C_1 , then we could merge S_{0, y_1} and C_1 using Theorem 3 to form a bigger cycle containing the square S_{0, m_1} in its interior. This is a contradiction since square S_{0, m_1} is in the interior of the cycle C_1 and by construction, the cycle C_1

is the outermost boundary cycle containing the square S_{0,m_1} in its interior (see property (c) of Lemma 6).

From the above paragraph, we therefore have that the square S_{0,y_1} lies in the interior of the cycle C_1 and $y_1 \geq m_2$. But by definition, the integer $m_2 - 2$ is the largest integer j such that $(0, j)$ lies in the interior of the cycle C_1 . Thus we have a contradiction and so every edge in $\Delta_1 \setminus \{e_1, g_1\}$ belongs to $R \setminus R_{left}$. An analogous proof as above holds for $y \leq m_1 - 2$.

To prove the final statement of (d1), suppose that some edge e in Δ_1 lies to the left of R_{left} . This necessarily means that $e = e_1$ or $e = g_1$. We suppose $e = g_1$ and arrive at a contradiction and an analogous proof holds for the other case. If $e = g_1$ lies to the left of R_{left} , then necessarily $e = g_1 = f_{NW}(0, m_2)$. This is because g_1 contains $(0, m_2 - 1)$ as an endvertex and the only edge lying to the left of R_{left} and containing $(0, m_2 - 1)$ as an endvertex is $f_{NW}(0, m_2)$. But this means that the occupied square S_{0,m_2} containing $f_{NW}(0, m_2)$ as an edge lies in the interior of the cycle C_1 . In particular, the vertex $(0, m_2) \in \mathcal{C} \cap R_{left}$ lies in the interior of the cycle C_1 , a contradiction to the definition of the integer m_2 (see paragraph following (5.2)). This proves (d1).

To prove property (d2), we proceed as follows. By the definition of the integers $\{m_i\}$, we have that the cycles C_i and C_{i+1} intersect at the vertex $(0, m_{i+1} - 1) \in R_{left}$. Using property (a3) of the previous subsection, we have that C_i and C_{i+1} intersect *only* at $(0, m_{i+1} - 1)$.

To prove the remaining part of (d2), we proceed as follows. We prove the statement regarding the edge $f_{SW}(0, m_i)$. An analogous proof holds for the other statement regarding the edge $f_{SE}(0, m_{i+1} - 2)$. Also we prove for $i = 2$ and an analogous statement holds for all $2 \leq i \leq h - 1$. The vertex $(0, m_2) \in \mathcal{C} \cap R_{left}$ lies in the interior of the cycle C_2 and so we have from property (b1) that the edge $f_{NW}(0, m_2)$ belongs to the cycle C_2 .

In Figure 7, the square S_{0,m_2} is represented by its centre $(0, m_2)$ and the edge $f_{NW}(0, m_2)$ is the line segment AB . If the edge $f_{SW}(0, m_2) = BD$ does not belong to the cycle C_2 , then it necessarily belongs to the interior of C_2 . Both the squares containing the edge $f_{SW}(0, m_2) = BD$ lie in the interior of the cycle C_2 . The square S_{1,m_2-1} with centre $(1, m_2 - 1)$ (represented by S_1 in Figure 7) also contains the edge $f_{SW}(0, m_2) = BD$ and therefore lies in the interior of C_2 .

The square S_{0,m_2-2} with centre $(0, m_2 - 2)$ (represented by S_2 in Figure 7) shares the edge $f_{NW}(1, m_2 - 1) = BC$ with the square S_{1,m_2-1} . Since the square S_{0,m_2-2} is occupied by definition, the square S_{0,m_2-2} also lies in the

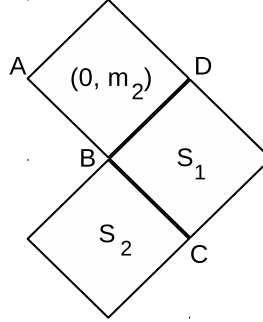


Figure 7: The square S_{0,m_2} is represented by the centre $(0, m_2)$. The squares S_1 and S_2 represent the squares S_{1,m_2-1} and S_{0,m_2-2} , respectively.

interior of the cycle C_2 . We prove this by contradiction. The cycle C_2 shares the edge $f_{NW}(1, m_2 - 1) = BC$ with the square S_{0,m_2-2} and so we can merge C_2 and S_{0,m_2-2} using Theorem 3 and obtain a bigger cycle containing the square S_{0,m_2} in its interior. This contradicts the fact that C_2 is the outermost boundary cycle containing the square S_{0,m_2} (see condition (c) of Lemma 6).

From the above paragraph, we therefore have that the square S_{0,m_2-2} lies in the interior of the cycle C_2 and by definition, the square S_{0,m_2-2} also lies in the interior of the cycle C_1 . Since the cycle C_1 and C_2 have mutually disjoint interiors (see property (a1)), we get a contradiction. This proves (d2).

To prove (d3), we consider the cycle graph H_{cyc} obtained as follows. Consider the vertex set $\{1, \dots, h\}$ and join i and j by an edge $e(i, j)$ if the corresponding cycles C_i and C_j share a common vertex.

The graph H_{cyc} constructed above is acyclic by property (3.1) in the proof of Theorem 1. Also, for every $1 \leq i \leq h - 1$, the cycles C_i and C_{i+1} intersect and so the vertices i and $i + 1$ are by the edge $e(i, i + 1)$ in H_{cyc} . Therefore the path $(e(1, 2), e(2, 3), e(3, 4), \dots, e(h - 1, h))$ is contained in the graph H_{cyc} . If the cycle C_2 shares a vertex with the cycle C_j for some $j \neq 1, 3$ then the vertices 2 and j would be also joined by an edge. This would mean that H_{cyc} contains a cycle, a contradiction. This proves (d3). ■

Outermost boundary with orientation

We recall that \mathcal{C} is the collection of vertices reachable by oriented open path starting from the left edge R_{left} of the rectangle R . We now introduce orientation only for the edges of the squares S_z for vertices $z \in \mathcal{C}$ and establish the properties needed for obtaining the oriented dual crossing. As before we denote the (dual) edges of the square S_z as $f_{NW}(z)$, $f_{SW}(z)$, $f_{NE}(z)$ and $f_{SE}(z)$. We assign the orientations \nearrow , \nwarrow , \searrow and \swarrow , respectively, to the edges $f_{NE}(z)$, $f_{NW}(z)$, $f_{SE}(z)$ and $f_{SW}(z)$ so that the edges in the square S_z form an oriented cycle.

For vertex $z \in \mathcal{C}$, we recall that the corresponding square S_z is defined to be occupied. An edge belonging to S_z can also belong to another occupied square S_w and therefore have two possible orientations. However from property (a2), we have that every dual edge in the outermost boundary $\cup_{i=1}^h C_i$ is a boundary edge adjacent to one occupied square and one vacant square. Therefore all dual edges in the outermost boundary have a unique orientation and we call them as *oriented dual* edges.

Henceforth unless otherwise mentioned, we consider only oriented dual edges.

We have the following properties regarding the orientation of the cycles C_i , $1 \leq i \leq h$, of the outermost boundary.

- (f1) For each i , $1 \leq i \leq h$, the subpath $(e_i, \Gamma_i, g_i) \subset C_i$ defined in (n1) – (n2) is an oriented subpath of C_i . Every edge in $C_i \setminus (e_i, \Gamma_i, g_i)$ has both endvertices in $R \setminus R_{left}$.
- (f2) For each i , $1 \leq i \leq h$, the cycle C_i is an oriented cycle; i.e., an oriented path with coincident endvertices.
- (f3) Suppose $h \geq 2$. For $1 \leq i \leq h - 1$, the edge $g_i = f_{SE}(0, m_{i+1} - 2)$ has orientation \searrow . For $2 \leq i \leq h$, the edge $e_i = f_{SW}(0, m_i)$ has orientation \swarrow .

The properties (f1) and (f3) are illustrated in Figure 8, where we have introduced orientation for the cycles C_1 and C_2 described in Figure 6. The oriented subpath (e_1, Γ_1, g_1) is shown where the oriented edge e_1 is denoted by the oriented segment zA , the oriented path $\Gamma_1 = ArstuvB$ and the oriented edge $g_1 = Bx$.

Proof of (f1) – (f3): We prove (f1) – (f2) and the first part of (f3) for $i = 1$ and an analogous proof holds for all i and the other cases.

The second statement of (f1) is true by the final statement in property (d1) and is used in the proof of (f2). To see that the first statement of (f1)

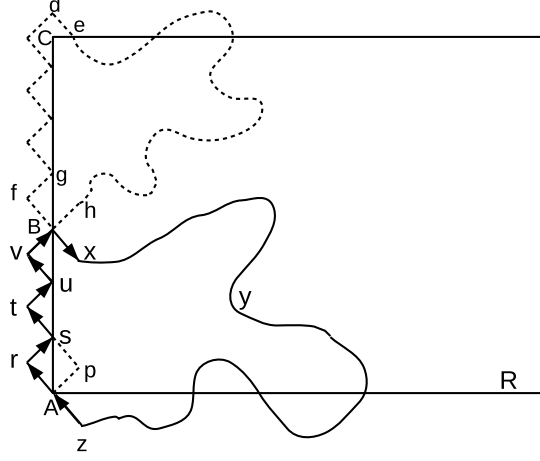


Figure 8: The cycles C_1 (thick line) and C_2 (dotted line) are shown meeting at the point B on the left edge of the rectangle R . Also the oriented subpath (e_1, Γ_1, g_1) of the cycle C_1 is illustrated with endvertices z and x .

is true, we recall from property (b1) that if $z \in \mathcal{C} \cap R_{left}$, then the edges $f_{NW}(z)$ and $f_{NE}(z)$ of the square S_z are consecutive edges in some cycle C_j of the outermost boundary. Also, the square S_z is occupied and the square S_{z_1} sharing the edge $f_{NW}(z)$ with S_z is vacant. Therefore $f_{NW}(z)$ has the orientation \nwarrow . Similarly, the edge $f_{NE}(z)$ has the orientation \nearrow . From property (d1), we have that Γ_1 is a subpath of the cycle C_1 . From the above, we obtain that Γ_1 is an *oriented* subpath of the cycle C_1 .

The edge e_1 defined in property (d1), shares an endvertex with the edge $f_{NW}(0, m_1) = f_{NW}(0, 1)$ of the cycle C_1 . Here we use $m_1 = 1$ (see definition of m_i prior to the properties (d1) – (d3)). There are therefore only two possibilities for e_1 . (p1) Either $e_1 = f_{SW}(0, 1)$ (represented by the segment Ap in Figure 8) or (p2) the edge $e_1 = f_{NW}(1, 0) = Az$. For the case of (p1), we argue as follows. From property (a2), the edge $e_1 = Ap$ is a boundary edge adjacent to one occupied square and one vacant square. The square $S_{0,1}$ (represented by $AA_1A_2A_3$ in Figure 8) with centre $(0, 1)$ (the midpoint of the segment As in Figure 8) is occupied and is contained in the interior of the cycle C_1 . Therefore the square $S_{1,0}$ with centre $(1, 0)$ sharing the edge $e_1 = f_{SW}(0, 1) = Ap$ with the square $S_{0,1}$ is vacant and lies in the exterior of C_1 . The edge $e_1 = Ap$ therefore has unique orientation \swarrow .

In the case of $(p2)$, the square $S_{0,-1}$ with centre $(0, -1)$ sharing the edge Az with $S_{1,0}$ is vacant by definition since the vertex $(0, -1)$ lies in the exterior of the rectangle R . Therefore the square $S_{1,0}$ with centre $(1, 0)$ is occupied and so the edge $e_1 = Az$ has unique orientation \nwarrow . In either case (e_1, Γ_1) is an oriented subpath of cycle C_1 . An analogous argument holds for the edge g_1 . This proves $(f1)$ and an analogous argument as above also proves the first statement of $(f3)$. An analogous argument holds for the second statement in $(f3)$.

To prove $(f2)$, we proceed by induction. Let $C_1 = (q_1, \dots, q_t)$ with $\{q_i\}$ being the edges of C_1 . From property $(f1)$, we assume that $q_1 = f_{NW}(0, 1)$, $q_2 = f_{NE}(0, 1)$, $q_3 = f_{NW}(0, 2)$, \dots , $q_{2m_2-3} = f_{NW}(0, m_2 - 2)$ and $q_{2m_2-2} = f_{NE}(0, m_2 - 2)$. Also the last edge $q_t = e_1$ and the edge $q_{2m_2-1} = g_1$ so that (e_1, Γ_1, g_1) is an oriented subpath of C_1 . In Figure 8, the edge $e_1 = Az$ and $g_1 = Bx$ and the oriented subpath (e_1, Γ_1, g_1) is shown with endvertices z and x .

For the induction step, suppose $(q_t, q_1, q_2, \dots, q_{k-1}, q_k)$ is an oriented path for some $2m_2 - 1 \leq k \leq t - 2$. We consider the case $q_k = f_{SE}(z)$ for some vertex z in the rectangle R with orientation \searrow . An analogous proof holds for the other three types of oriented edges. We define the neighbouring squares of S_z as follows. Let S_{z_1} and S_{z_3} be the squares sharing the edges $f_{SE}(z)$ and $f_{SW}(z)$, respectively with the square S_z . Let S_{z_2} be the square that shares a corner with S_z and share edges with S_{z_1} and S_{z_3} . This is illustrated in Figure 9(a) where the square S_z is shown along with the squares $S_{z_i}, i = 1, 2, 3$. The label i corresponds to the square S_{z_i} , for $i = 1, 2, 3$. The oriented segment AB represents the edge $q_k = f_{SE}(z)$.

We use the following property in the proof.

The square S_z is occupied and lies in the interior of the cycle C_1 .

The square S_{z_3} is vacant and lies in the exterior of C_1 . (5.4)

Proof of (5.4): The edge $q_k = f_{SE}(z) = f_{NW}(z_3)$ is common to both the squares S_z and S_{z_3} . From property $(a2)$, we have that every edge in the cycle C_1 is a boundary edge adjacent to one occupied square contained in the interior of C_1 and one vacant square in the exterior of C_1 . In particular, one of the squares S_z or S_{z_3} is vacant and the other is occupied. If S_{z_3} is occupied, then the orientation of q_k would be \nwarrow , a contradiction. Thus S_z is occupied and S_{z_3} is vacant and this proves (5.4). ■

We now determine the endvertex common to the edges q_k and q_{k+1} . By

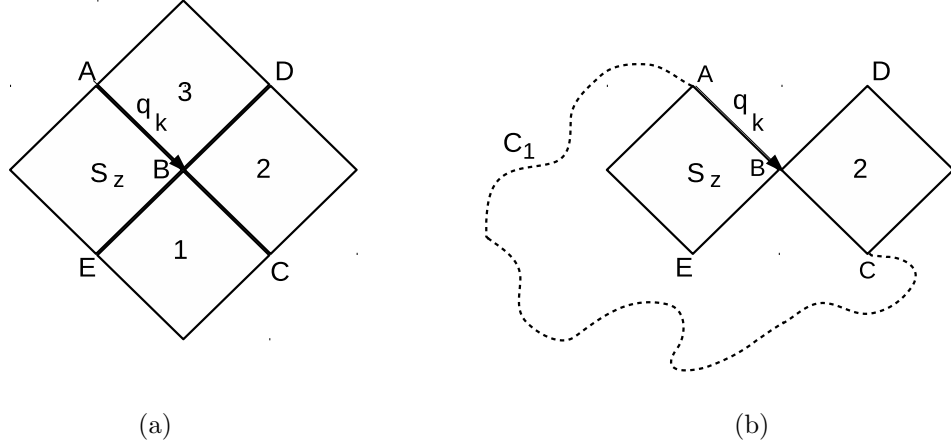


Figure 9: (a) Squares neighbouring S_z . The label i corresponds to square S_{z_i} for $i = 1, 2, 3$. (b) In case (x3), exactly one of the squares S_z or S_{z_2} lies in the interior of C_1 .

induction assumption, we have that the edges (q_{k-1}, q_k) form a consistent pair and so the non arrow endvertex A of the edge q_k (see Figure 9) is coincident with the arrow endvertex of q_{k-1} . The vertex A has degree two in the cycle C_1 and so the edge q_{k+1} does not contain A as an endvertex. This means that the edge q_k shares the arrow endvertex B of the edge q_k and so the possible choices for the edge q_{k+1} are $f_{SW}(z)$, $f_{SE}(z_1)$ and $f_{NE}(z_2)$, represented by the segments BE , BC and BD , respectively.

We consider three cases: (x1) the squares S_{z_1} and S_{z_2} are both vacant, (x2) the square S_{z_1} is occupied and S_{z_2} is vacant and (x3) the square S_{z_2} is occupied. In each case, we see that the edges q_k and q_{k+1} form a consistent pair.

Consider first the case (x1). The edge q_{k+1} is not $f_{SE}(z_1) = BC$ since the latter edge is adjacent to two vacant squares S_{z_1} and S_{z_2} . We recall that every edge in the cycle C_1 is a boundary edge belonging to an occupied square and a vacant square. Similarly, the edge q_{k+1} is not $f_{NE}(z_2) = BD$ since the latter is again adjacent to two vacant squares (see (5.4)). Thus $q_{k+1} = f_{SW}(z) = BE$ with the corresponding orientation \swarrow .

The argument for case (x2) is analogous as in the previous paragraph. The edge q_{k+1} cannot be $f_{SW}(z) = BE$ since the latter belongs to two occupied squares S_z and S_{z_1} (see (5.4)). Also q_{k+1} cannot be $f_{NE}(z_2) = BD$ since the latter belongs to two vacant squares S_{z_2} and S_{z_3} (again using 5.4).

Therefore $q_{k+1} = f_{SE}(z_1) = BC$ with the corresponding orientation \searrow .

For case (x3), we use the property (a3) that the cycles $\{C_i\}$ in the outermost boundary have mutually disjoint interiors and share at most one vertex in common. Moreover, the endvertex common to C_i and C_{i+1} belongs to the left edge R_{left} of R . The square S_{z_2} is occupied and shares a vertex $B \notin R_{left}$ with the occupied square S_z contained in the interior of the cycle C_1 . The square S_{z_2} therefore also belongs to the interior of the cycle C_1 . Here $B \notin R_{left}$ since the edge q_{k+1} belongs to the subpath $C_1 \setminus (e_1, \Gamma_1, g_1)$ and therefore has both its endvertices in $R \setminus R_{left}$ using property (f1).

If the edge $q_{k+1} = f_{NW}(z_2) = f_{SE}(z_1)$, then the cycle C_1 encloses exactly one of the squares S_z or S_{z_2} in its interior. This is illustrated in Figure 9(b) where the segment AB is the edge $q_k = f_{SE}(z)$ and the segment $BC = f_{NW}(z_2)$. Therefore, $q_{k+1} \neq f_{NW}(z_2)$ and so $q_{k+1} = f_{NE}(z_2)$ with the orientation \nearrow . This completes the induction step for the case $q_k = f_{SE}(z)$ for some vertex $z \in \mathcal{C}$ with orientation \searrow . An analogous proof holds for the other three types of oriented edges. \blacksquare

6 Proof of Theorem 5

In this section, we prove the mutual exclusivity of open oriented left right and closed dual oriented top bottom dual crossings. Let $LR_{or}(R)$ be the event that there exists an open oriented left right crossing and let $TD_{or}^*(R)$ be the event that there is a closed dual oriented top bottom crossing of the rectangle R . As in the unoriented case, the above two events cannot occur together. Suppose not and let Π_L be an open oriented left right crossing and let Π_D^* be a closed dual oriented top bottom crossing.

Every edge in Π_L lies in the interior of the rectangle R in the sense that if $e \in \Pi_L$ then no endvertex of e lies in the exterior of R . Also, since every edge in Π_D^* is closed, every edge in Π_D^* also lies in the interior of R (see the definition of closed dual edges two paragraphs prior to statement of Theorem 5). Therefore the paths Π_L and Π_D^* intersect in the sense that there are edges $e \in \Pi_L$ and $f \in \Pi_D^*$ that intersect. The edge e belonging to the original percolation model of Figure 2 is open and the dual edge f intersecting e is closed, a contradiction to the definition of openness of the dual edges.

If the event $LR_{or}(R)$ does not occur, we obtain a closed dual oriented top

bottom crossing in a two step procedure described below.

Concatenating the paths $\{\Delta_i\}_{1 \leq i \leq h}$ to form Δ_{tot}

The first step is to concatenate the paths $\{\Delta_i\}$ defined in properties (d1) – (d3). We recall the definition of the integers $\{m_i\}$ defined in the previous subsection (see (5.2)). We also recall that m and n are the width and the height of the rectangle R . We have the following properties.

(y1) Fix $1 \leq i \leq h$. The path Δ_i has endvertices $v_{i+1} = (0, m_{i+1} - 1)$ and $(0, m_i - 1)$. For $i \neq j$, the paths Δ_i and Δ_j are edge disjoint. At least one edge in the path Δ_h intersects the top edge R_{top} of the rectangle R and at least one edge in the path Δ_1 intersects the bottom edge R_{bottom} of R .

(y2) Let $\Delta_{tot} = \cup_{i=1}^h \Delta_i$. We have that Δ_{tot} is an oriented path with endvertices $(0, m_{h+1} - 2)$ and $(0, m_1 - 1) = (0, 0)$. If $\Delta_{tot} = (h_1, \dots, h_r)$, then the edge $h_1 \in \Delta_h$ contains the endvertex $(0, m_{h+1} - 2)$ and the edge $h_r \in \Delta_1$ contains the endvertex $(0, 0)$. Moreover there are indices $k_1 < k_2$ such that h_{k_1} intersects the top edge R_{top} of the rectangle R and h_{k_2} intersects the bottom edge R_{bottom} of R .

(y3) Let $w = (x, y)$ be an endvertex of a (dual) edge $e \in \Delta_{tot}$. If w lies in the exterior of R , then the other endvertex of e intersects the boundary of the rectangle R . If R has no oriented left right crossing in the original percolation model (see Figure 2(a)), then $0 \leq x \leq m$ and $-1 \leq y \leq n + 1$.

Proof of (y1) – (y3): We prove (y3), (y1) and (y2) in that order. For (y3), we argue as follows. Let $w = (x, y)$ be an endvertex of an edge $e \in \Delta_i$. If the vertex w lies in the interior of the rectangle R , then $0 \leq x \leq m$ and $0 \leq y \leq n$.

If the vertex w lies in the exterior of the rectangle R , then the other endvertex w_1 of the edge $e \in \Delta_i$ necessarily intersects the boundary of R . The endvertex w_1 cannot lie in the interior of R because this means that w either belongs to the boundary or lies in the interior of R . If w_1 does not belong to the boundary of R and lies in the exterior of R , then the square S_z with centre z and containing e as an edge lies in the exterior of R . This leads to a contradiction since all dual edges belong to squares $S_z, z \in \mathcal{C}$ and no vertex in the oriented cluster \mathcal{C} lies in the exterior of R (see first subsection of Section 5). Thus the endvertex w_1 intersects the boundary of R and the bounds on the y -coordinate for the vertex w holds.

To obtain the bounds on the x -coordinate, we recall that e is an edge

of the square S_z with centre $z \in \mathcal{C}$. Since the rectangle R has no left right crossing, the vertex z does not belong to the right edge R_{right} of R . Thus both the endvertices of e lie to the left of R_{right} and so $x \leq m$. To see $x \geq 0$, we use property (d1) that every edge in Δ_i lies to the right of R_{left} . This proves the first statement of (y3). The argument in the previous paragraph also proves the second statement of (y3).

The first statement of (y1) is true from property (d1) since the path Δ_i has the same endvertices $(0, m_{i+1} - 1)$ and $(0, m_i - 1)$ as the path $\Gamma_i = C_i \setminus \Delta_i$.

For the second statement of (y1), we use property (a3) of the outermost boundary. The cycles C_i and C_j of the outermost boundary are edge disjoint for $i \neq j$. Since the path $\Delta_i \subset C_i$, we obtain the second statement of (y1).

For the third statement of (y1), we argue as follows. From the first statement of (y1), the path Δ_1 has $(0, m_1 - 1)$ as an endvertex. Since $m_1 = 1$ (see (5.2)), the path Δ_1 contains the origin as endvertex and so the second half of the third statement of (y1) is true.

To prove the first half, we again use the first statement of (y1) that the path Δ_h contains $(0, m_{h+2} - 1)$ as an endvertex. Since $m_{h+2} - 1 \geq n$ (see (5.3)), the endvertex v_{h+1} either lies above the top edge R_{top} or touches the top edge of the rectangle R . Suppose endvertex v_{h+1} lies above R_{top} . Arguing as in the proof of (y3), we obtain that the other endvertex w_{h+1} of the dual edge $g_h \in \Delta_h$ containing v_{h+1} intersects the top edge R_{top} . This proves the first half of the final statement of (y1) and therefore completes the proof of (y1).

To prove (y2), we have from property (f2) that every $\Delta_i, 1 \leq i \leq h$ is an oriented subpath of the cycle C_i . Therefore if $h = 1$, then $\Delta_{tot} = \Delta_1$ is an oriented path. Suppose $h = 2$ and consider the two subpaths Δ_1 and Δ_2 .

Using property (d2), we have that Δ_1 and Δ_2 intersect only at the endvertex $(0, m_2 - 1)$. Thus $\Delta_1 \cup \Delta_2$ is a path. To see that it $\Delta_1 \cup \Delta_2$ an oriented path, we argue as follows. From property (f3) we have that the last edge $e_2 \in \Delta_2$ is $f_{SW}(0, m_2)$ with orientation \swarrow . Also the first edge $g_1 \in \Delta_1$ is $f_{SE}(0, m_2 - 2)$ with orientation \searrow . From property (d1) we have that both the edges e_2 and g_1 share the endvertex $(0, m_2 - 1)$. In particular, the arrow endvertex of e_2 coincides with non arrow endvertex of g_1 . Thus (e_2, g_1) is a consistent pair and we have that $\Delta_1 \cup \Delta_2$ is an oriented path.

Suppose $h \geq 3$. Arguing as above, we have that $\Delta_1 \cup \Delta_2$ is an oriented path. Consider now the union $\Delta_1 \cup \Delta_2 \cup \Delta_3$. Using property (d2), we have that the path Δ_3 intersects the path Δ_2 at the point $(0, m_3 - 1)$. Arguing as in the previous paragraph, we have that $\Delta_3 \cup \Delta_2$ is an oriented path. Using

property (d3), we also have that Δ_3 does not intersect the path Δ_1 . Thus $\Delta_3 \cup \Delta_2$ intersects Δ_1 only at the vertex $(0, m_2 - 1)$ and so $\Delta_3 \cup \Delta_2 \cup \Delta_1$ is also an oriented path. Continuing iteratively, we have that $\Delta_{tot} = \cup_{i=1}^h \Delta_i$ is an oriented path with endvertices $(0, m_{h+1} - 2)$ and $(0, m_1 - 1) = (0, 0)$. This proves the first statement of (y2).

To prove the second statement of (y2) we argue as follows. We have from property (y1) that the path Δ_h contains $(0, m_{h+1} - 2)$ as an endvertex. Also no other path $\Delta_i, 1 \leq i \leq h - 1$ contains $(0, m_{h+1} - 2)$ as an endvertex. Thus $h_1 \in \Delta_h$ contains the endvertex $(0, m_{h+1} - 2)$. Similarly the edge $h_r \in \Delta_1$ contains the endvertex $(0, m_1 - 1)$ and since $m_1 = 0$ (see (5.2)), this proves the second statement of (y2). The final statement in (y2) follows from property (y1). \blacksquare

Extracting the dual crossing from Δ_{tot}

Let $\Delta_{tot} = (h_1, \dots, h_r)$. Using property (y2) above, let j_1 be the largest index $j < k_2$ such that the edge h_j intersects the top edge of R . Similarly, again using property (y2), let j_2 be the smallest index $j > j_1$ such that h_j intersects the bottom edge of R . We have the following properties.

- (z1) The edge h_{j_1} touches the top edge of R and has orientation \searrow or \swarrow .
- (z2) The edge h_{j_2} touches the bottom edge of R and has orientations \searrow or \swarrow .
- (z3) If R has no oriented crossing in the original oriented percolation model of Figure 2(a), then every edge in $\Delta_{cr} = (h_{j_1}, h_{j_1+1}, \dots, h_{j_2})$ lies in the interior of R .

Thus the path Δ_{cr} is the desired dual oriented top bottom crossing.

Proof of (z1)–(z3): We prove (z3) first. Suppose some edge $h_k \in \Delta_{cr}, k \geq j_1$ lies in the exterior of the rectangle R . From property (y3) above, we have that at least one endvertex of h_k intersects the boundary of R . Suppose h_k intersects the top edge of R . We arrive at a contradiction as follows. Let v_k and v_{k+1} be the non arrow and arrow endvertices of the edge h_k , respectively. Suppose that one of v_k or v_{k+1} lies in the exterior of the rectangle R . If the non arrow endvertex v_k lies in the exterior of R , the edge h_{k+1} sharing the non arrow endvertex v_k also intersects the top edge of R . This contradicts the definition of the index j_1 .

If the arrow endvertex v_{k+1} of the edge h_k lies in the exterior of R , then the edge h_{k+1} also lies in the exterior of R and the arrow endvertex of h_{k+1}

intersects the top edge of R . This again contradicts the definition of j_1 and so the edge h_k does not touch the top edge of R .

An analogous proof holds if h_k intersects the bottom edge of R and lies in the exterior of R . If the edge h_k intersects the left edge R_{left} of R and lies in the exterior of R , then h_k lies to the left of R_{left} . But since $h_k \in \cup_{i=1}^h \Delta_i$, this contradicts the final statement of property (d1). Finally suppose h_k lies to the right of R_{right} and let x_k and x_{k+1} be the x -coordinates of the vertices v_k and v_{k+1} , respectively. We must have either $x_k \geq m+1$ or $x_{k+1} \geq m+1$, a contradiction to property (y3). Thus every edge in Δ_{cr} lies in the interior of R and this proves .

Using an argument analogous to the first two paragraphs of proof of (z1) we obtain that h_{j_1} has orientation \searrow or \swarrow . This proves (z1) and an analogous proof holds for h_{j_2} in the property (z2). ■

Proof of Theorem 5: It only remains to see that every edge in Δ_{cr} is closed and we prove this as follows. Let e be an oriented dual edge and let S_a and S_b be the squares with centres a and b , respectively, containing the edge e . Here a, b are vertices in the rectangle R . The edge e belongs to some cycle C_i of the outermost boundary and so from property (a2), we have that one of the squares in $\{S_a, S_b\}$ is occupied and the other is vacant. We recall that $S_z, z \in R$ is occupied if the vertex z belongs to the oriented cluster \mathcal{C} defined in the second paragraph of Section 5. Thus one of the vertices of $\{a, b\}$ belongs to \mathcal{C} and the other does not belong to \mathcal{C} . Therefore the edge f joining a and b is closed and so the edge e intersecting the edge f is also closed (see two paragraphs prior to statement of Theorem 5). ■

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